Lorentzian Algebra for the Superstring

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A Lorentzian algebra is constructed that underlies the covariant formulation of the spinning string. It is a simply laced hyperbolic Kac-Moody algebra having a realization on the ghost-extended spectrum of string states. From the weight space of this algebra, we obtain a unique indefinite superalgebra that has a realization on the superstring spectrum and that automatically implements the Gliozzi-Scherk-Olive projection.

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One of the major quests in string theory is the search for symmetries underlying its remarkable properties. An important advance is the Frenkel-Kac realization[1] of simply laced finite and affine Lie algebras in terms of bosonized vertex operators. Among other applications, this construction relates the spectrum of any ten-dimensional string to the weight lattice of an associated Lie algebra.

In the covariant formulation,[2] bosonization of the fermionic ghost naturally leads to a Lorentzian extension of the weight lattice.[3,4] This suggests the existence of a larger algebra, which for the superstring would incorporate covariant supersymmetry. In this Letter, we identify a particular Lorentzian algebra that is a key structure underlying string spectra. We also find a related indefinite superalgebra relevant to the superstring case.

Since the weight lattice is Lorentzian, we need an extension of the Frenkel-Kac construction. Consider an arbitrary integral indefinite lattice $\Gamma^{p,q}$ in $R^{p,q}$ with metric signature $(+)^p(-)^q$, generated by the $p+q$ basis vectors e^μ. Introduce $p+q$ two-dimensional free massless scalar fields $\phi^\mu(z)$. Then, to each lattice point $\alpha=\alpha_\mu e^\mu$, associate an infinite tower of operators

$$E^\alpha_{(\zeta)}=\oint\frac{dz}{2\pi i}e^\alpha_{(\zeta)}(z):=\oint\frac{dz}{2\pi i}\partial^{(\zeta)}\phi(z)e^{\alpha\cdot\phi}(z)c_\alpha, \quad (1)$$

where

$$\partial^{(\zeta)}\phi(z)=\prod_{i=1}^{r}\partial^{\zeta_i}\phi^{\mu_i}(z),\quad r,\zeta_i\in N,$$
$$\partial^{(0)}\phi(z):=1. \quad (2)$$

The cocycle operators c_α are given by

$$c_\alpha=\exp\left[i\pi\left(\alpha\cdot M\oint\frac{dz}{2\pi i}\partial\phi\right)\right],$$
$$M^\mu_\nu=\begin{cases}0, & \mu\le\nu,\\ \pm 1, & \mu>\nu.\end{cases} \quad (3)$$

For convenience, the Lorentz indices $\mu_1,\ldots,\mu_r$ associated to $(\zeta)=(\zeta_1,\ldots,\zeta_r)$ are suppressed. Normal ordering is understood throughout.

Eventually, we wish to identify the operators $E^\alpha_{(\zeta)}$ as vertex operators in a string theory. To do so, several constraints must be imposed. Conformal invariance relates $|\zeta|=\sum_i\zeta_i$ to α and selects certain linear combinations of the $E^\alpha_{(\zeta)}$. Furthermore, only a subset of the lattice is admissible because of physical constraints; examples of such constraints are $m_L^2=m_R^2$ and the restriction to specified spinor ghost sectors. Note also that the operators $E^\alpha_{(\zeta)}$ with α and $|\zeta|$ fixed are linearly dependent through the relations

$$\oint\frac{dz}{2\pi i}\partial f(\partial\phi,e^{\alpha\cdot\phi})=0 \quad (4)$$

for arbitrary f.

The complete set $\{E^\alpha_{(\zeta)}\}$ may be given an algebraic structure by invoking operator-product expansions:

$$\partial^{(\zeta)}\phi(z)e^{\alpha\cdot\phi}(z)\partial^{(\eta)}\phi(w)e^{\beta\cdot\phi}(w)\sim\sum_{n=\alpha\cdot\beta-|\zeta|-|\eta|}^{\infty}(z-w)^n{\sum_{(\theta)}}'c^{n(\theta)}_{(\zeta)(\eta)}(\alpha,\beta)\partial^{(\theta)}\phi(w)e^{(\alpha+\beta)\cdot\phi}(w). \quad (5)$$

As before, Lorentz indices are suppressed; the prime indicates the restriction $|\theta|=|\zeta|+|\eta|-\alpha\cdot\beta+n$. Then, since the contour integrals select $n=-1$, we find for the nonvanishing graded commutators

$$[E^\alpha_{(\zeta)},E^\beta_{(\eta)}\}=\epsilon(\alpha,\beta){\sum_{(\theta)}}'c^{-1(\theta)}_{(\zeta)(\eta)}(\alpha,\beta)E^{\alpha+\beta}_{(\theta)}, \quad (6)$$

where

$$[E^{\alpha},E^{\beta}\}:=E^{\alpha}E^{\beta}-(-1)^{\alpha^2\beta^2}E^{\beta}E^{\alpha}, \tag{7}$$

and

$$\begin{aligned}&\epsilon(\alpha,\beta):=\exp[i\pi(\alpha\cdot M\beta)],\\ &|\theta|=|\zeta|+|\eta|-\alpha\cdot\beta-1.\end{aligned} \tag{8}$$

The choice (3) for the cocycle operators results in a superalgebra Z_2 grading in Eq. (6): Operators from even and odd points obey commutation and anticommutation rules, respectively.[5]

The structure of the algebra (6) may be partially understood as vector addition on the lattice, $\alpha+\beta\in\Gamma^{p,q}$. Note that the conformal dimensions of the operators in Eq. (6), given for $E^{\alpha}_{(\zeta)}$ by $h_{\alpha}=\frac{1}{2}\alpha(\alpha+Q)+|\zeta|-1$ for some background charge Q, satisfy $h_{\alpha}+h_{\beta}=h_{\alpha+\beta}$. The subalgebra V of invariant or vertex operators, which has $h=0$, plays a dominant role. As any such operator is a conserved charge, the string states at any given mass level fall into representations of V. In the following, we focus on V alone.

A Fock-space representation of the algebra (6) provides the link to the string spectrum. The vacuum $|0\rangle$ obeys $a^{\mu}_n|0\rangle=0$ for $n\geq 0$, where the a^{μ}_n are coefficients in the mode expansion

$$\partial\phi^{\mu}(z)=\sum_{n=-\infty}^{+\infty}a^{\mu}_n z^{-n-1}.$$

The state $|\alpha,(\zeta)\rangle$ is created from the vacuum by action of the integrand of Eq. (1) at $z=0$:

$$|\alpha,(\zeta)\rangle=e^{\alpha}_{(\zeta)}(0)|0\rangle. \tag{9}$$

The algebra V is a direct generalization of the standard vertex-operator realization of Lie algebras. As usual, the Cartan subalgebra of V is spanned by the momentum operator $E^{0}_{(1)}=a_0\equiv p$:

$$[p^{\mu},E^{\alpha}_{(\zeta)}]=\alpha^{\mu}E^{\alpha}_{(\zeta)}, \tag{10}$$

$$[E^{\alpha}_{(\zeta)},E^{-\alpha}_{(\eta)}\}=\epsilon(\alpha,-\alpha)c^{-1(1)}_{(\zeta)(\eta)}\cdot p\,\delta_{\alpha^2+|\zeta|+|\eta|,2}. \tag{11}$$

Finite-dimensional subalgebras of V occur only for α restricted to a Euclidean sublattice ($q=0$) and $\alpha^2\leq 2$, as may be seen from the graded commutators

$$[E^{\alpha}_{(0)},E^{\beta}_{(0)}\}=\epsilon(\alpha,\beta)\times\begin{cases}0, & \alpha\cdot\beta\geq 0,\\ E^{\alpha+\beta}_{(0)}, & \alpha\cdot\beta=-1,\\ \frac{1}{2}(\alpha-\beta)\cdot E^{\alpha+\beta}_{(1)}, & \alpha\cdot\beta=-2,\\ -\frac{1}{2}(\alpha\times\beta)\cdot E^{\alpha+\beta}_{(1,1)}, & \alpha\cdot\beta=-3,\\ \ldots\end{cases} \tag{12}$$

Thus, for $\Gamma^{p,0}$ we obtain simply laced Lie algebras and superalgebras.[6] In contrast, the bosonized covariant formulation of the spinning string requires $q=1$. The resulting algebras, called Lorentzian, are indefinite Kac-Moody algebras[7] or their superalgebra generalizations.

Modular invariance of string theories requires that the relevant lattices be self-dual.[8] In the Lorentzian case, such lattices are determined by their dimension, $p+1$. The only even self-dual Lorentzian lattices are

$$\Pi^{4n+1,1}:=\{\alpha\text{ or }\alpha-\lambda\in Z^{4n+1,1}\,|\,\alpha\cdot\lambda\in Z\}, \tag{13}$$

where λ is a constant vector with components $\lambda^{\mu}=\frac{1}{2}$, and n is even. Furthermore, any odd self-dual Lorentzian lattice is isomorphic to $Z^{p+1,1}$, modulo a Lorentz rotation.[9]

An example important for string theory is the odd self-dual lattice $\Pi^{4n+1,1}$, where n is *odd*. This lattice can be so$(4n+1,1)$-rotated into $Z^{4n+1,1}$. Note, however, that these two lattices can lead to physically inequivalent theories because the condition $h=0$ is not Lorentz invariant in the presence of a background charge Q.

Both $\Pi^{4n+1,1}$ and $Z^{4n+1,1}$ are embedded in the nonintegral lattice

$$\begin{aligned}\Lambda^{p,1}&=\{\alpha\text{ or }\alpha-\lambda\in Z^{p,1}\},\quad p=4n+1,\\ &=(o)\cup(v)\cup(s)\cup(c),\end{aligned} \tag{14}$$

as the conjugacy classes $(o)\cup(s)$ and $(o)\cup(v)$, respectively. The conjugacy-class addition table and scalar products are identical to those of so$(8n)$. This larger lattice can be interpreted as the weight lattice of a Lorentzian algebra, $D_{p,1}$, whose root lattice is the conjugacy class (o). The latter is spanned by the simple roots α^i of so$(2p)\equiv D_p$ and by an extra root α^0:

$$\begin{aligned}&\alpha^i=e^i-e^{i+1},\quad i=1,\ldots,p-1,\\ &\alpha^p=e^{p-1}+e^p,\\ &\alpha^0=e^0-e^1-e^2-e^3.\end{aligned} \tag{15}$$

The corresponding Dynkin diagram is shown in Fig. 1. The algebra may also be obtained from the affine algebra $D^{(1)}_{p-1}$ by addition to the root lattice of a second null direction.

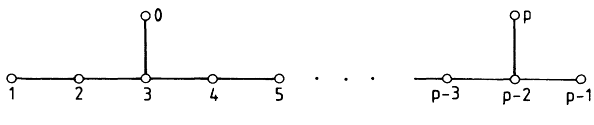

FIG. 1. Dynkin diagram for $D_{p,1}$.

For the ten-dimensional string theories with $p=5$, $q=1$, $\Gamma^{5,1}$ should be the minimal Lorentzian extension of the weight lattice of $\mathrm{so}(10)\equiv D_5$, which is $\Lambda^{5,1}$. The associated Lorentzian algebra is $D_{5,1}$, whose Dynkin diagram, shown in Fig. 2, exhibits a triality symmetry because the elementary vector and spinor weights have the same length. It is the unique simple simply laced Lorentzian extension of so(10) and is a hyperbolic Kac-Moody algebra.[10]

The algebra $D_{5,1}$ governs the space-time part of the ghost-extended spinning-string spectrum, as may be seen from its elementary weights and the fields that create them from the vacuum:

$$\begin{aligned}
\lambda_o &= (1\,0\,0\,0\,0,\ -1) \sim \psi_{(-1)},\\
\lambda_v &= (0\,0\,0\,0\,0,\ -1) \sim T_{(-1)},\\
\lambda_s &= \tfrac{1}{2}(1\,1\,1\,1\,1,\ -3) \sim S_{(-3/2)},\\
\lambda_c &= \tfrac{1}{2}(1\,1\,1\,1\,-1,\ -3) \sim \dot{S}_{(-3/2)}.
\end{aligned} \tag{16}$$

Here, ψ is the Neveu-Schwarz-Ramond field, T is the tachyon of the bosonic string, and $S,\dot{S}$ are spin fields of opposite helicity. The subscript in parentheses indicates the ghost charge q.[4]

It is intriguing that $D_{5,1}$ automatically yields the correct correlation between the ghost charges and the so(10) conjugacy classes. Moreover, with $q=-1$ in the Neveu-Schwarz sector and $q=-\frac{3}{2}$ in the Ramond sector, the algebra naturally selects one of the two canonical choices for the spinor ghost sectors.

Note that to obtain an integral lattice $\Lambda^{5,1}$ must be projected either to $Z^{5,1}$ or to $\Pi^{5,1}$. The former possibility restricts the spectrum to the Neveu-Schwarz sectors of the spinning string and is possible in any even dimension. The latter yields the superstring and is possible only for dimensionality $8n+2$.

The role of supersymmetry has not yet been directly addressed. One approach is to find an integral sublattice of $\Lambda^{5,1}$ that plays the role of a root lattice for some superalgebra. This may be achieved by extending the root lattice of $D_{5,1}$ by an odd weight in the (v), (s), or (c) conjugacy classes. Since a minimal simple extension is desired, any roots that become nonsimple must be removed. The procedure is analogous to the construction of E_8 from the so(16) root lattice by addition of the conjugacy class (s).

Addition of a vector weight e^5 and removal of α^5 results in a root lattice $Z^{5,1}$, while addition of a spinor weight $-2e^0-\lambda_s$ and removal of α^0 yields $\Pi^{5,1}$. The associated superalgebra $E_{5,1}$ is the unique simple simply laced extension of $D_{5,1}$. It can be minimally represented by the Dynkin-Kac diagram[11] of Fig. 3.

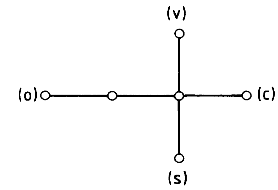

FIG. 2. Dynkin diagram for $D_{5,1}$.

Supersymmetry generators are invariant operators associated to odd points α with $\alpha^2\leq 1$. For the Neveu-Schwarz sectors of the spinning string, $(x)=(v)$, the anticommuting generators in the algebra $E_{5,1}$ have $h\in Z+\frac{1}{2}$ and so are not conserved. They change the level number N by odd half-units and map Gliozzi-Scherk-Olive (GSO)-accepted states into GSO-rejected ones. For the superstring, $(x)=(s)$, the anticommuting generators have $h\in Z$ since $Q=2e^0$. The subset corresponding to lattice points satisfying $\frac{1}{2}\alpha^2-q\leq 1$ is a set of conserved supersymmetry charges. All generators commute with the GSO projection.

Note that $D_{5,1}$ and $E_{5,1}$ are the invariant subalgebras V of the full lattice algebras associated with $Z^{5,1}$ and $\Pi^{5,1}$, respectively.

The maximal finite subsuperalgebra of $E_{5,1}$ is the semidirect sum of a Grassmann algebra with su(5). For the superstring, this results in a partial symmetry on the spectrum. No finite superalgebra exists with a nontrivial action on the whole superstring spectrum. This suggests that an off-shell realization of supersymmetry will involve an infinite number of auxiliary fields.

In addition to their intrinsic interest as extensions of the usual string-spectrum symmetry in which the ghost-extended spin fields arise naturally, the lattice algebras $D_{5,1}$ and $E_{5,1}$ have other consequences. For example, the picture-changing operation,[2] which relates different slices of the weight lattice of $E_{5,1}$, is expected to have an algebraic realization. More computationally useful applications could include the use of the Ward identities of $D_{5,1}$ to compute correlations by the reduction technique.[12] Furthermore, enlargements of the algebras may be significant. Extension to the heterotic strings[8] could yield further insights on their origin and relationships, while inclusion of the space-time operators ∂X might

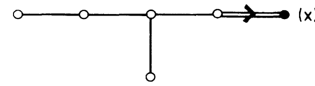

FIG. 3. Dynkin-Kac diagram for $E_{5,1}$. The symbol (x) represents the possible conjugacy classes (v), (s), or (c) in $D_{5,1}$ of the labeled root.

lead to an improved understanding of the nature of space-time supersymmetry.

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