PULLBACK OF CURRENTS BY MEROMORPHIC MAPS

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My dissertation researches on properties of iterations $f^n = f \circ f \circ \ldots \circ f$ (n times) of a selfmap $f : X \to X$. Here X is a compact Kähler manifold and f is a dominant meromorphic map. For holomorphic maps, a variational principle for smooth maps proves the existence of a measure which is invariant under f and has maximal entropy (i.e. the entropy of the measure equals the topological entropy). The same question is harder to answer for a general meromorphic map f, due to the fact that f is not continuous. Since the pioneer work of Bedford et al. on Hénon maps, a common strategy is to first establish the existence of appropriate invariant currents (a generalization of measures), and then use them to construct invariant measures. To this end, it is important to know what currents can be pulled back or pushed forward by a map f. In my dissertation, based upon a regularization theorem of Dinh and Sibony, I give a definition of pulling back by a given meromorphic map for a large class of currents. This pullback operator is compatible with the definitions given by many other authors. Many applications and examples are given.

Contents

CHAPTER 1

Introduction

The purpose of this dissertation is to define a meaningful pullback operator by meromorphic maps for currents. The definition so defined has desired good properties, is compatible with the definitions in previous works, and has applications to many examples of complex dynamical systems.

Given X a compact Kähler manifold of dimension k and $f : X \to X$ a dominant meromorphic map, complex dynamics studies dynamical objects associated to f such as topological entropy, invariant measures, the distribution of periodic points,...

In the case $k = 1$, X is a complex curve. Then any meromorphic map $f : X \to X$ is automatically a holomorphic map (because the indeterminacy set of f has codimension ≥ 2 in X, and hence is empty since X has dimension 1). The theory of smooth maps on compact manifolds ensures the existence of an invariant measure of maximal entropy of f (see Newhouse [**64**]). Recall that a measure μ is invariant by f if the pushforward $f_*(\mu)$ equals μ. When the topological degree of f is greater than 1 (i.e. when f is not an automorphism), a dynamical construction of the (unique) invariant measure of maximal entropy can be given in terms of repelling periodic points of f (see Brolin [**18**], Freie -Lopes - Mane [**53**] and Lyubich [**62**]).

Going up one dimension to the case $k = 2$, we face many difficulties which can be illustrated by the examples of Hénon maps (which are among the most extensively studied maps in two and higher dimensional, see e.g. Hubbard - Oberste Vorth [**59**][**60**]). These are maps of the form $f : \mathbb{C}^2 \to \mathbb{C}^2$, $f(x, y) = (y, p(y) - \delta x)$, where p is a polynomial of degree ≥ 2 and δ is a non-zero complex constant. These are automorphisms of $\mathbb{C}^2$, and we can lift them to birational maps on $\mathbb{P}^2 \supset \mathbb{C}^2$. Since the topological degree of a Hénon map f is 1, the construction of an invariant measure μ of maximal entropy can not be directly done as in dimension 1. Bedford-Lyubich-Smillie [**7**][**8**][**9**][**10**] instead proceeded as follows: They first construct invariant positive closed $(1,1)$ currents T^+ and T^- for f and f^{-1}, and the measure μ is defined as the wedge product $T^+ \wedge T^-$. We see two obstacles needed to be overcome: 1) We need to explain how to pullback positive closed $(1,1)$ currents and measures by non-smooth maps; and 2) We need to define the wedge product of two positive closed $(1,1)$ currents. The pullback of positive closed $(1,1)$ currents can be done by making use of their local potentials, which are pluri-subharmonic functions (see Meo [**63**]). The local potentials of T^+ and T^- on $\mathbb{C}^2$ are continuous, hence the theory of Bedford-Taylor [**11**] can be used to define the wedge product of them on $\mathbb{C}^2$. It turns out that T^+ and T^- are regular enough, so we can define their wedge product even on $\mathbb{P}^2$. Moreover, using dynamical properties, the resulting measure μ can be shown to have no mass on pluri-polar sets. Therefore, μ can be pulled back by any meromorphic map; and we can check that it is an invariant measure for the Hénon map in question.

There are now many works on constructing invariant measures and invariant positive closed $(1,1)$ currents (the so called Green currents) under variant dynamical constraints, both in two and higher dimensions. (The following is only a small portion of the current literature: [**4**], [**6**], [**14**], [**15**], [**16**], [**17**], [**20**], [**21**], [**22**], [**25**], [**26**], [**27**], [**28**], [**29**], [**30**], [**31**], [**32**], [**33**], [**34**], [**36**], [**37**], [**40**], [**41**], [**42**], [**44**], [**45**], [**47**], [**48**], [**49**], [**50**], [**51**], [**52**], [**56**], [**57**], [**65**], [**67**], [**68**].) As in the case of Hénon maps, a common strategy to construct invariant measures for a dominant meromorphic maps consists of the following steps: 1) construct appropriate invariant positive closed currents T^+ and T^- of complement degrees for the pullback and pushforward of f; 2) construct the measure μ as the wedge product $T^+ \wedge T^-$; and 3) show that the resulting measure μ has no mass on pluri-polar sets, and then check that it is an invariant measure. All of these three steps impose very challenging technical difficulties, and there are several recent works addressing one or several of these issues (see Dinh-Sibony [**42**][**43**][**44**][**45**][**46**]). Among these three steps, Step 1 is most relevant to the topic of this dissertation, so we will discuss it in some detail in the below.

When $k \geq 3$, one of the currents T^+ and T^- in Step 1 must be of bidegree higher than $(1,1)$, and pulling back such a current by a meromorphic map may be problematic. In contrast to the case of bidegree $(1,1)$, it is not always able to pullback a positive closed current of higher bidegrees even by a holomorphic map: If $\pi : X \to Y$ is the blowup of a three-fold Y along a smooth curve $C \subset Y$, what can be the pullback by the map π of the current of integration on C? Hence we can see that understanding what currents can be pulled back by a given meromorphic map is useful for constructing invariant measures and more generally invariant currents of that map.

There are several works on pulling back currents of higher bidegrees (see Allesandrini -Bassnelli [**2**], the cited papers of Dinh and Sibony, and Russakovskii - Shiffman [**67**]). In these definitions, if T is a positive closed current which can be pulled back by a meromorphic map f, then the resulting current $f^*(T)$ is again positive closed. Such a conservation of positivity is expected under some meaningful dynamical constraints on f, for example when f has one dynamical degree strictly greater than the others. (However, there are many interesting examples of pseudo-automorphisms on blowups of $\mathbb{P}^3$ whose first and second dynamical degrees are the same (see Bedford - Kim [**5**], Perroni - Zhang [**66**] and Blanc [**12**]); and this phenomenon of having no strictly dominant dynamical degree may be prevalent for pseudo-automorphisms on these manifolds as evidenced by the case of automorphisms, see [**70**].) Let us illustrate that this conservation of positivity fails for the following very simple pseudo-automorphism on blowups of $\mathbb{P}^3$. Let $J : \mathbb{P}^3 \to \mathbb{P}^3$ be the map $[x_0 : x_1 : x_2 : x_3] \to [1/x_0 : 1/x_2 : 1/x_3 : 1/x_4]$. Let $\pi : X \to \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ at 4 points $e_0 = [1 : 0 : 0 : 0]$, $e_1 = [0 : 1 : 0 : 0]$, $e_2 = [0 : 0 : 1 : 0]$ and $e_3 = [0 : 0 : 0 : 1]$. Let $J_X = \pi^{-1} \circ J \circ \pi$ be the lift of J to X. Let $\Sigma_{i,j} \subset \mathbb{P}^3$ be the line connecting e_i and e_j, and let $\widetilde{\Sigma_{i,j}}$ be its strict transform in X. We see that in cohomology $J_X^*\{\widetilde{\Sigma_{0,1}}\} = -\{\Sigma_{2,3}\}$. Therefore, whatever a meaningful pullback of the current of integration $[\widetilde{\Sigma_{0,1}}]$ we give for the map J_X, the resulting $J_X^*[\widetilde{\Sigma_{0,1}}]$ can not be a positive current.

The goal of my dissertation is to define a meaningful pullback operator for meromorphic maps, so that it is both compatible with previous definitions and applicable to "abnormal" cases like the map J_X. The main idea is to use duality and regularization of currents in defining pullback. We consider the more general setting of $f : X \to Y$ a dominant meromorphic map between compact Kähler manifolds X and Y. Let T be a (p, p) current on Y. Assume that we can pullback T by f, and the resulting $f^*(T)$ is a (p, p) current on X. Then for a smooth form α on X of appropriate bidegrees, we should have

$$\int_X f^*(T) \wedge \alpha = \int_Y T \wedge f_*(\alpha).$$

(The pushfoward $f_*(\alpha)$ of a smooth form α is well-defined for any meromorphic map f. However, it may not be smooth, even when f is holomorphic.) Thus the mysterious current $f^*(T)$ is understood if we can make sense of the action of the current T on the currents $f_*(\alpha)$ for all smooth forms α. Since Y is compact, the current T is of a finite order s. Since the current $f_*(\alpha)$ is DSH, we can use a regularization result of Dinh and Sibony to produce approximations of $f_*(\alpha)$ by C^s forms with good properties. This enables us to define $\int_Y T \wedge f_*(\alpha)$ using limits.

The details of the definition including the notion of DSH currents will be given in the subsequent chapters. We close this chapter by explaining what we obtain when applying the definition to our friend the map J_X. The "critical set" of the map J_X has dimension 1, hence for any positive closed $(2,2)$ current T on X, $J_X^*(T)$ is well-defined. Moreover, $J_X^*[\widetilde{\Sigma_{0,1}}] = -[\widetilde{\Sigma_{2,3}}]$, which is compatible to the pullback on cohomology level. Even more specially, the map J_X enjoys what we call "2-analytic stable". This means that for any positive closed $(2,2)$ current T, $(J_X^*)^2(T) = (J_X^2)^*(T) = T$. The latter property enables us to construct invariant $(2,2)$ currents for the map J_X.

CHAPTER 2

Preliminary results

This chapter presents some fundamental notions and results of complex analysis, differential geometry and complex dynamics used in the main chapter. Most of the results will be stated without proofs or with sketches of the proofs only, but references will be given for the convenience of the readers.

1. d and d^c operators on complex manifolds

The main references for this section is Chapter 1 in the book of Demailly [**23**] and Chapter 0 in the book of Griffiths - Harris [**54**].

Let $\mathbb{C}$ be the complex plane and let $\mathbb{C}^n$ be the n-dimensional complex space. We use complex coordinates $z_1, z_2, \ldots, z_n$ for $\mathbb{C}^n$. If we write $z_j = x_j + iy_j$ where $x_j, y_j \in \mathbb{R}$ and $i^2 = -1$ then $x_1, y_1, \ldots, x_n, y_n$ comprises real coordinates for $\mathbb{C}^n$.

Let Ω be an open set in $\mathbb{C}^n$. Then Ω is a complex manifold of dimension n and is a real manifold of dimension $2n$. At a point in Ω, the tangent vector space T_Ω has a natural structure of the complex vector space $\mathbb{C}^n$. We denote by $T_\Omega^{\mathbb{R}}$ the underlying real tangent space. It admits $(\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_n, \partial/\partial y_n)$ as a basis. The almost complex structure J on $T_\Omega^{\mathbb{R}}$ is given by $J(\partial/\partial x_k) = \partial/\partial y_k$ and $J(\partial/\partial y_k) = -\partial/\partial x_k$ for $k = 1, \ldots, n$.

The complexified tangent space $\mathbb{C} \otimes T_\Omega = \mathbb{C} \otimes_{\mathbb{R}} T_\Omega^{\mathbb{R}} = T_\Omega^{\mathbb{R}} \oplus iT_\Omega^{\mathbb{R}}$ admits a basis consisting of vectors

$$\frac{\partial}{\partial z_k} = \frac{1}{2}(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}),\ \frac{\partial}{\partial \overline{z_k}} = \frac{1}{2}(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k})$$

for $k = 1, 2, \ldots, n$. Note that

$$\begin{aligned} Id_{\mathbb{C}} \otimes J(\frac{\partial}{\partial z_k}) &= i\frac{\partial}{\partial z_k}, \\ Id_{\mathbb{C}} \otimes J(\frac{\partial}{\partial \overline{z_k}}) &= -i\frac{\partial}{\partial \overline{z_k}}. \end{aligned}$$

The vector subspace generated by the vectors $\partial/\partial z_k$ is denoted by $T_{\Omega}^{1,0}$ (holomorphic vectors or vectors of type $(1, 0)$), and the vector subspace generated by the vectors $\partial/\partial\overline{z_k}$ is denoted by $T_{\Omega}^{0,1}$ (anti-holomorphic vectors or vectors of type $(0, 1)$).

We then have a canonical decomposition $\mathbb{C} \otimes T_{\Omega} = T_{\Omega}^{1,0} \oplus T_{\Omega}^{0,1} \sim T_{\Omega} \otimes \overline{T_{\Omega}}$ via the isomorphisms

$$\begin{aligned} \xi \in T_{\Omega} &\mapsto \frac{1}{2}(\xi - iJ\xi) \in T_{\Omega}^{1,0}, \\ \xi \in \overline{T_{\Omega}} &\mapsto \frac{1}{2}(\xi + iJ\xi) \in T_{\Omega}^{0,1}. \end{aligned}$$

Here $\overline{T_{\Omega}}$ has complex structure $-J$ and thus is conjugate to T_{Ω}.

By duality, we have a corresponding decomposition for (complex valued) 1-forms on Ω:

$$Hom_{\mathbb{R}}(T_{\Omega}^{\mathbb{R}}, \mathbb{C}) \equiv Hom_{\mathbb{C}}(\mathbb{C} \otimes T_{\Omega}^{\mathbb{R}}, \mathbb{C}) \equiv T_{\Omega}^{*} \oplus \overline{T_{\Omega}^{*}}.$$

More specifically, $(dx_1, dy_1, \ldots, dx_n, dy_n)$ is a basis for $Hom_{\mathbb{R}}(T_{\Omega}^{\mathbb{R}}, \mathbb{C})$, $(dz_1, \ldots, dz_n)$ is a basis for T_{Ω}^{*}, and $d\overline{z_1}, \ldots, d\overline{z_n}$ is a basis for $\overline{T_{\Omega}^{*}}$. If $f : \Omega \to \mathbb{C}$ is a C^1 function then we can write

$$df = \sum_{k=1}^{n}(\frac{\partial f}{\partial x_k}dx_k + \frac{\partial f}{\partial y_k}dy_k) = \sum_{k=1}^{n}(\frac{\partial f}{\partial z_k}dz_k + \frac{\partial f}{\partial \overline{z_k}}d\overline{z_k}).$$

A map f is called a holomorphic map if $\partial f/\partial\overline{z_k} = 0$ for every $k = 1, \ldots, n$. From the above formula, f is holomorphic iff df is $\mathbb{C}$-linear.

Now we discuss a generalization of splitting the exterior differential d acting on forms of higher degrees. The complexified exterior algebra $\mathbb{C}\otimes_{\mathbb{R}}\Lambda_{\mathbb{R}}^{*}(T_{\Omega}^{\mathbb{R}})^{*} = \Lambda_{\mathbb{C}}^{*}(\mathbb{C}\otimes T_{\Omega}^{*})$

is decomposed as follows:

$$\Lambda^k(\mathbb{C}\otimes T_\Omega)^* = \Lambda^k(T_\Omega \oplus \overline{T_\Omega})^* = \bigoplus_{p+q=k} \Lambda^{p,q}T^*_\Omega.$$

Here

$$\Lambda^{p,q}T^*_\Omega = \Lambda^p T^*_\Omega \otimes \Lambda^q \overline{T^*_\Omega},$$

consisting of smooth differential forms of bidegree (or type) (p,q). For $s \geq 0$ and an open subset U of Ω, we denote by $C^s(U, \Lambda^{p,q}T_\Omega)$ the class of C^s forms of type (p,q) on U. Such a form can be written as

$$u(z) = \sum_{|I|=p,|J|=q} u_{I,J}(z) dz_I \wedge d\overline{z_J}.$$

Here $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ are multiple indices, $dz_I = d_{z_{i_1}} \wedge \ldots \wedge d_{z_{i_p}}$ and $d\overline{z_J} = d\overline{z_{j_1}} \wedge \ldots \wedge d\overline{z_{j_q}}$, and $u : U \to \mathbb{C}$ is a function of class C^s. The exterior $d : \Lambda^k T^*_\Omega \to \Lambda^{k+1}T^*_\Omega$ splits into $d = d' + d"$, where $d' : \Lambda^{p,q}T^*_\Omega \to \Lambda^{p+1,q}T^*_\Omega$ and $d" : \Lambda^{p,q}T^*_\Omega \to \Lambda^{p,q+1}T^*_\Omega$ are given by:

$$\begin{aligned} d'u &= \sum_{I,J} \sum_{1\leq k\leq n} \frac{\partial u_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z_J}, \\ d"u &= \sum_{I,J} \sum_{1\leq k\leq n} \frac{\partial u_{I,J}}{\partial \overline{z_k}} d\overline{z_k} \wedge dz_I \wedge d\overline{z_J} \end{aligned}$$

for a smooth differential form $u = \sum_{|I|=p,|J|=q} u_{I,J}(z) dz_I \wedge d\overline{z_J}$.

The identity $0 = d^2 = (d' + d")^2 = (d')^2 + d'd" + d"d' + (d")^2$ implies by taking into account the bidegrees that $0 = (d')^2 = (d")^2 = d'd" + d"d'$.

We define $d^c = \frac{1}{2\pi i}(d' - d")$. Then d^c is a real operator, i.e. $\overline{d^c u} = d^c \overline{u}$ for smooth forms u. Moreover, $dd^c = \frac{i}{\pi} d'd"$.

These considerations can be developed for complex manifolds. Recall that a complex manifold X of dimension n is a differentiable manifold equipped with a holomorphic atlas τ_α. This means that there is a covering U_α of X, open sets $V_\alpha \subset \mathbb{C}^n$, and homeomorphisms $\tau_\alpha : U_\alpha \to V_\alpha$ such that the transition maps

$$\tau_{\alpha,\beta} = \tau_\alpha \circ \tau_\beta^{-1} : \tau_\beta(U_\alpha \cap U_\beta) \to \tau_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic maps. A (local) (p,q) form on U_α is given by a (p,q) form u_α of type (p,q) on V_α. Two local (p,q) forms u_α and u_β are compatible if we have the identity $u_\beta = \tau^*_{\alpha,\beta}(u_\alpha)$ on the intersection $U_\alpha \cap U_\beta$. If two local forms u_α and u_β are compatible, then they define a form u on the larger set $U_\alpha \cup U_\beta$, and this construction can be extended to define forms on any open set U of X.

2. Currents

2.1. Currents on differentiable manifolds. The main references for this subsection is the books de Rham [**24**] and Demailly [**23**].

a) Currents:

Let X be a smooth differentiable manifold of real dimension n. Let $\Omega \subset X$ be a coordinate open set, i.e. Ω is diffeomorphic to an open subset of $\mathbb{R}^n$, with coordinates $x_1, \ldots, x_n$. Let u be a smooth p form on Ω, then we can write $u(x) = \sum_{|I|=p} u_I(x) dx_I$. Here $I = (i_1, \ldots, i_p)$ is a multi-index, $u_I(x)$ is a smooth function, and $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$. To a compact set $L \subset \Omega$ and an integer s we define a seminorm

$$p_L^s(u) = \sup_{x \in L} \max_{|\alpha| \leq s} |D^\alpha u(x)|.$$

That is, the seminorm is defined as supremum on L of all the derivatives D^α of multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $|\alpha| = m \leq s$.

On the space $\mathcal{E}^p(X)$ of global smooth p forms on X, we equip the topology defined by all semi-norms p_L^s when s, L, and Ω vary. Then we define $\mathcal{D}^p(X)$ to be the subspace of $\mathcal{E}^p(X)$ consisting of smooth forms of compact supports.

A current T of dimension p (or degree n-p) is a linear functional on $\mathcal{D}^p(X)$ so that for each subspace $K \subset\subset X$ then the restriction of T to $\mathcal{D}^p(K)$ is continuous. If T is a current of dimension p and φ is a test p form then we denote by $\langle T, \varphi\rangle$ the action of T on φ. We also define the action of T on φ by the notation $\int_X T \wedge \varphi$.

Similarly, we can equip the space $\mathcal{E}_s^p(X)$ of C^s forms with the semi-norms p_L^s where L and Ω vary, and then let $\mathcal{D}_s^p(X)$ to be the subspace of C^s forms with compact support. Then a current T that acts continuously on $\mathcal{D}_s^p(X)$ is called a current of order s.

LEMMA 2.1. *Let X be a compact manifold, and let T be a current on X. Then T is of finite order, i.e. there is an integer $s \geq 0$ so that T is of order s.*

PROOF. (Sketch) Assume otherwise. Then there are smooth forms φ_j on X so that $||\varphi_j||_{C^j} = 1$ and $|\langle T, \varphi_j\rangle| \geq j$ for all $j = 1, 2, \ldots$. By Arzela-Ascoli theorem, there is a smooth form φ which is the limit point of a fixed subsequence of φ_j in any C^k topology. Since T is a current, it follows that the finite number $|\langle T, \varphi\rangle|$ is the limit of the corresponding subsequence of $|\langle T, \varphi_j\rangle|$, and the latter is infinite. This is a contradiction.

□

On the space of currents, we equip the weak topology. Hence if T_n and T are currents, we say that T_n converges to T and denote by $T_n \rightharpoonup T$ if for any test form φ then $\langle T_n, \varphi\rangle$ converges to $\langle T, \varphi\rangle$.

Example 1: If T is a q form whose coefficients are locally L^1, then T defines a current of dimension $n - q$ (and of degree q) as follows: If φ is a test form then

$$\langle T, \varphi\rangle := \int_X T \wedge \varphi.$$

The current defined this way is of order 0, i.e. it can act on continuous forms.

Example 2: If Z is an oriented closed submanifold of dimension p of X, then we define the current of integration $[Z]$ of dimension p as follows: If φ is a test form then

$$\langle [Z], \varphi \rangle := \int_Z \varphi|_Z.$$

Then $[Z]$ is a current of order 0.

b) Wedge product and exterior derivative of currents:

Wedge product: If T is a current of degree p and ψ is a smooth form of degree q, then we define the wedge product $T \wedge \psi$ to be the current of degree $p+q$ which acts on test forms φ as follows: $\langle T \wedge \psi, \varphi \rangle := \langle T, \varphi \wedge \psi \rangle$.

Exterior derivative: If T is a current of dimension q and of order s, then we define dT to be the current of dimension $q-1$ and of order $s+1$ as follows: If u is a C^{s+1} test form of degree $q-1$ then $\langle dT, u \rangle := (-1)^{n-q+1} \langle T, du \rangle$. A current T is closed if $dT = 0$.

Example 1: If f is a C^1 form then the exterior derivatives of f, both as a C^1 form and as a current, are the same.

Example 2: If Z is an oriented submanifold of dimension p of X with boundary ∂Z then by Stokes theorem we see that $d[Z] = (-1)^{n-p+1}[\partial Z]$.

Example 3: If T is a distribution then it is a current of degree zero.

c) Push-forward (or direct image) of currents by proper maps: Let $f : X \to Y$ be a smooth map between smooth manifolds. Let T be a current on X of dimension p and of order s. Assume that the restriction of f to $Supp(T)$ is proper, i.e. for any compact set $K \subset Y$ then $f^{-1}(K) \cap Supp(T)$ is compact. Then we define the push-forward (or direct image) $f_*(T)$, which is a current on Y of dimension p and order s, as follows: If φ is a test form then $\langle f_*(T), \varphi \rangle := \langle T, f^*(\varphi) \rangle$.

Remark that the push-forward and the exterior derivative are commutative to each other, i.e. $d(f_*(T)) = f_*(dT)$.

d) Pull-back of currents by submersions: Assume that $f : X \to Y$ is a submersion, i.e. f is surjective and the derivative $d_x f : T_{X,x} \to T_{Y,f(x)}$ is surjective as well. For a current T on Y of degree p and of order s, we define $f^*(T)$ as a current on X of degree p and of order s as follows: $\langle f^*(T), \varphi\rangle := \langle T, f_*(\varphi)\rangle$ for test forms φ.

e) Trivial extension of currents of order zero

PROPOSITION 2.1. *Let $x_1, \ldots, x_n$ be a coordinate system of an open set $\Omega \subset X$. Every current T on X of degree q and of order s can be written in a unique way as*

$$T = \sum_{|I|=q} T_I dx_I$$

on Ω, where T_I are distributions of order s on Ω.

PROOF. (Sketch, see Proposition 2.9 Chapter 1 in Demailly's book) The T_I are defined as follows: Let J be a multi-set with $|J| = n - q$ so that its underlying set is the complement of I in the set $\{1, 2, \ldots, n\}$, and $dx_I \wedge dx_J$ defines the same orientation as $dx_1 \wedge \ldots \wedge dx_n$. Then for a smooth function f on Ω of compact support we define

$$\langle T_I, f\rangle := \langle T, f dx_J\rangle.$$

□

Now let $C \subset X$ be a closed subset, and let $X' = X - C$. Let T be a current on X' of degree q and of order zero. Then by Proposition 2.1, locally we can write

$$T = \sum_{|I|=q} T_I dx_I,$$

where T_I are distribution of order zero, and hence are complex-valued measures by Riesz representation theorem. Assume now that all the T_I has locally finite mass near C. Then we can trivially extend these measures to measures $\widetilde{T}_I$ defined on X, by declaring $\widetilde{T}_I(A) = T_I(A - C)$ for a Borel set $A \subset X$. The current, which defined locally as

$$\widetilde{T} = \sum_{|I|=q} \widetilde{T}_I dx_I$$

is a well-defined current on X of degree q and of order zero. By definition, if φ is a smooth form of compact support contained in $X - C$, then $\langle \widetilde{T}, \varphi \rangle = \langle T, \varphi \rangle$. We call $\widetilde{T}$ the trivial extension of T across C.

2.2. Currents on complex manifolds. On a complex manifold, Lelong defined the notion of positive currents, which are very useful in complex dynamics. The main references are the books Lelong [**61**] and Demailly [**23**].

a) Positive currents:

Let X be a complex manifold of dimension n. A smooth (p,p) form v on X is said to be strongly positive if locally we can write

$$v(z) = \sum_s \gamma_s(z) i\alpha_{s,1}(z) \wedge \overline{\alpha_{s,1}}(z) \wedge \ldots \wedge i\alpha_{s,p}(z) \wedge \overline{\alpha_{s,p}}(z),$$

where $\gamma_s(z) \geq 0$ and $\alpha_{s,j}(z)$ are $(1,0)$ forms.

A smooth (p,p) form u on X is said to be positive if for any strongly positive $(n-p, n-p)$ then $u \wedge v$ is a positive (n,n) form (i.e. a positive measure).

Kähler manifold: Let X be a complex manifold. A Kähler $(1,1)$ form ω on X is a strictly positive closed $(1,1)$ form ω. That is, ω is positive, ω is nowhere zero, and $d\omega = 0$. If X has a Kähler form then X is a Kähler manifold.

A (p,p) current T is positive if for any strongly positive test form α then $\langle T, \alpha \rangle \geq 0$. Likewise, a (p,p) current T is strongly positive if for any positive test form then $\langle T, \alpha \rangle \geq 0$.

LEMMA 2.2. *A positive current T is a current of order* 0.

PROOF. (Sketch) By definition, we need to show that for any compact set $K \subset X$, there is a constant $C_K > 0$ so that for any smooth form φ on X with support in K

then $|\langle T,\varphi\rangle| \leq C_K||\varphi||_{L^\infty}$. By using a partition of unity, we may assume that K is contained in a coordinate open set $\Omega \subset X$. Let $z_1,\ldots,z_n$ be a coordinate system on Ω. Consider a non-negative smooth function λ on X so that support of λ is contained in Ω, and $\lambda = 1$ on an open neighborhood of K. Then there is a constant $C > 0$ independent of φ so that $C||\varphi||_{L^\infty}\omega^q \pm \varphi$ are strongly positive smooth forms, here $\omega = \lambda(idz_1 \wedge d\bar{z}_1 + \ldots + idz_n \wedge d\bar{z}_n)$ and $(n-q,n-q)$ is the bidegree of T. Since T is a positive current, we have $|\langle T,\varphi\rangle| \leq C_K||\varphi||_{L^\infty}$, where $C_K = C\langle T,\omega^q\rangle$. □

If X is in addition a compact Kähler manifold with a Kähler $(1,1)$ form ω_X, then we define the mass of a positive (p,p) current T as $||T|| := \langle T,\omega_X^{n-p}\rangle$. Similarly, if T is a negative current (i.e $-T$ is a positive current) then we define the mass of T to be the mass of $-T$.

b) Positive closed currents-Lelong numbers: If a (p,p) current T is both positive and closed (i.e. $dT = 0$) then we say that it is a positive closed current. Note that if T is a positive closed current, then the mass of T (defined above) depends only on the cohomology class of T.

We can assign to a positive closed current T its Lelong number, which can be defined locally as follows: Let X be an open subset in $\mathbb{C}^n$ and let $z_1,\ldots,z_n$ be coordinates of $\mathbb{C}^n$. For a point $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$ we define $|z|^2 = |z_1|^2 + \ldots |z_n|^2$ the square of its Euclidean norm. Let T be a positive closed current of bidimension (p,p) on X. For $a \in X$, we define the Lelong number $\nu(T,a)$ of T at a by the following formula

$$\nu(T,a) = \lim_{r\to 0}\frac{1}{r^{2p}}\int_{|z-a|<r} T\wedge(\frac{i}{2\pi}d'd"|z|^2)^p.$$

Example 1: If φ is a local L^1 function on X which is upper-semicontinuous and so that $dd^c\varphi \geq 0$, then we say that φ is a pluri-subharmonic function (or PSH for short). When φ is a PSH function then $T = dd^c\varphi$ is a positive closed $(1,1)$ current.

Its Lelong numbers can be computed by the formula

$$\nu(T,x)=\liminf_{z\to x}\frac{\varphi(z)}{\log|z-x|}.$$

Example 2: Let Z be an irreducible subvariety of dimension p of X. and let Z_{reg} be the regular set of Z. Then Z defines a strongly positive closed current of bidimension (p,p), called the current of integration on Z, denoted by $[Z]$ as follows: If φ is a smooth test form then

$$<[Z],\varphi>:=\int_{Z_{reg}}\varphi|_{Z_{reg}}.$$

If $x\in X-Z$ then $\nu([Z],x)=0$, while if $x\in Z$ then $\nu([Z],x)=$ the multiplicity of Z at x (Thie's theorem).

Proof. (Sketch, see Theorem 7.7 page 168 in Demailly's book) Use the comparison theorems for Lelong's numbers. □

c) DSH currents: DSH currents were introduced by Dinh and Sibony to assist the study of complex dynamics. We briefly recall the definition here, please see [**39**] for more details.

Let X be a compact Kähler manifold. A (p,p) current T is DSH if there are positive (p,p) currents T_1 and T_2, and for some positive closed $(p+1,p+1)$ currents $\Omega_i^{\pm}$ (for $i=1,2$) so that $T=T_1-T_2$ and $dd^cT_i=\Omega_i^+-\Omega_i^-$. For a DSH (p,p) current T, we define its DSH norm as

$$||T||_{DSH}=\min\{||T_1||+||T_2||+||\Omega_1^+||+||\Omega_2^+||\}$$

where the minimum is taken on all currents $T_1,T_2,\Omega_1^{\pm},\Omega_2^{\pm}$ as above.

We say a sequence of DSH (p,p) currents T_n converge to a current T in DSH if T_n weakly converges to T in the sense of currents and $||T_n||_{DSH}$ is uniformly bounded.

2.3. Support theorems for currents. Let X be a complex manifold.

a) Support theorem for normal currents: We say that a current T is normal if both T and dT are of order 0.

THEOREM 2.1. *Let T be a normal current of bidimension (p,p). If support of T is contained in an analytic set A of dimension $< p$ then $T = 0$.*

PROOF. (Sketch, see Theorem 2.10 Chapter 3 in Demailly's book [**23**])

The regular part A_{reg} of A is a complex submanifold of $X - A_{sing}$ of dimension $< p$. The current $T_{reg} = T|_{X-A_{sing}}$ has support in A_{reg}. The problem being local, we consider a coordinate open set Ω of $X - A_{sing}$. Let $g_1, \dots, g_m$ be real C^1 functions on Ω such that $A_{reg} \cap \Omega = \{x \in \Omega :\ g_1(x) = \ldots = g_m(x) = 0\}$ and $dg_1 \wedge \ldots \wedge dg_m \neq 0$ on Ω. The rank of the matrix $(d'g_k)$ is at least $n-p+1$ at any point of $A_{reg} \cap \Omega$. Therefore we can choose a continuous frame $(\zeta_1, \dots, \zeta_n)$ of $(1,0)$ forms on Ω so that at least $n-p+1$ of them are in the set $(d'g_k)$. Then $g_k T|_{X-A_{sing}} = g_k \wedge d'T|_{X-A_{sing}} = 0$ on Ω. It follows that $d'g_k \wedge T|_{X-A_{sing}} = 0$. From this we can conclude that $T|_{X-A_{sing}} = 0$, which implies that T has support in A_{sing}.

Now A_{sing} has smaller dimension than A, we may use the above argument with A replaced by A_{sing} and induction on the dimension of A to deduce that $T = 0$ on X. □

b) Siu's decomposition theorem: Let T be a positive closed (p,p) current on X. If $c > 0$ then the set $E_c(T) = \{x \in X :\ \nu(T,x) \geq c\}$ is an analytic subset of X of codimension $\geq p$ (it may be an empty set).

PROOF. See Siu's paper [**69**] and Theorem 8.16 Chapter 3 in the book of Demailly [**23**]. □

In particular, there are (at most countable) irreducible subvarieties V_j of codimension p, positive numbers λ_j, and a positive closed (p,p) current R for which $E_c(R)$

has codimension $> p$ for all $c > 0$, so that we have a decomposition

$$T = R + \sum_j \lambda_j [V_j].$$

c) $\mathbb{C}$-normal currents: The class of $\mathbb{C}$-normal currents was studied by Bassanelli [**3**]. A current T is $\mathbb{C}$-normal if T and $dd^c T$ are both of order zero. Hence the class of $\mathbb{C}$-normal currents includes the class of DSH currents. As such, $\mathbb{C}$-normal currents are very useful in complex dynamics.

The class of $\mathbb{C}$-normal currents is contained in the class of more general objects called $\mathbb{C}$-flat currents. A current T is $\mathbb{C}$-flat if locally it can be written as $T = F + \partial G + \overline{\partial} H$ where F, G, H are L^1_{loc} currents.

Bassanelli proved the following results:

THEOREM 2.2. *Let T be a $\mathbb{C}$-flat current of bidimension (p, p). If support of T has Hausdorff $2p$-dimension zero then $T = 0$.*

PROOF. (Sketch, see Theorem 1.13 in the cited Bassanelli's paper.) The main idea to consider the projection of ϕT to complex subspaces of dimension p where ϕ is a smooth function of compact support, using that under these projections then the image of a $\mathbb{C}$-flat current is again $\mathbb{C}$-flat. □

PROPOSITION 2.2. *Let C be a closed subset of X and let T be a $\mathbb{C}$-flat current on $X - C$ with locally finite mass across C. Then its trivial extension T^0 is $\mathbb{C}$-flat on X.*

PROOF. (Sketch, see Proposition 1.22 in the cited paper of Bassanelli) Let ϕ_n be an increasing sequence of smooth functions such that $0 \leq \phi_n \leq 1$, $\phi_n = 0$ in a neighborhood of C and the limit of ϕ_n is 1 in $\Omega - C$. Then show that T^0 is the limit of $\phi_n T$ in the mass norm. □

3. Cohomology groups

Let X be a compact Kähler manifold. Please see the books of de Rham [**24**], Griffiths-Harris [**54**] and Demailly [**23**] for more detail.

3.1. de Rham cohomology. Recall that a smooth form φ is closed if $d\varphi = 0$, and it is exact if $\varphi = d\psi$ for some smooth form ψ. The de Rham p-th cohomology of X is defined as

$$H^p_{DR}(X) = \frac{\{\text{closed smooth } p \text{ forms}\}}{\{\text{exact smooth } p \text{ forms}\}}.$$

de Rham theorem says that de Rham cohomology $H^p_{DR}(X)$ is the same as the singular cohomology $H^p(X)$.

Proof. (Sketch, see page 44 in the book of Griffiths and Harris): Let $\mathcal{A}^q$ be the sheaf of smooth q forms on X, and let $\mathbb{R}$ be the sheaf of locally constant functions on X. Then by Poincare lemma for the operator d, we have a LES:

$$0 \to \mathbb{R} \to \mathcal{A}^0 \to \mathcal{A}^1 \dots \tag{1}$$

here each map in the sequence is the exterior differential d. Then the singular cohomology is the Cech cohomology of the constant sheaf $\mathbb{R}$. The latter is the same as the de Rham cohomology of X, as can be seen via the SES's

$$0 \to \mathbb{R} \to \mathcal{A}^0 \to \mathcal{Z}^1 \to 0, ...$$

which are deduced from the above LES. □

Similarly, if in the definition of de Rham cohomology we use the sheaves of currents on X, then Poincare lemma for currents can be used to prove its isomorphism to the singular cohomology.

3.2. Dolbeault cohomology. Similarly as above we define the notations of $\overline{\partial}$ closed and exact forms and currents. Then we define Dolbeault cohomology groups as follows:

$$H^{p,q}_{\overline{\partial}}(X) = \frac{\{\overline{\partial} \text{ closed smooth } (p,q) \text{ forms}\}}{\{\overline{\partial} \text{ exact smooth } (p,q) \text{ forms}\}}.$$

Dolbeault theorem identifies the Dolbeault cohomology group $H^{p,q}(X)$ and the sheaf cohomology $H^q(X,\Omega^p)$. Here Ω^p is the sheaf of holomorphic p-forms. The proof is similar to the proof of de Rham theorem (see page 45 in the book of Griffiths and Harris), using Poincare lemma for the $\overline{\partial}$ and the LES

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \to \mathcal{A}^{p,1} \to \ldots,$$

here $\mathcal{A}^{p,q}$ is the sheaf of smooth (p,q) forms on X.

Similarly, Dolbeault cohomology can be computed using currents instead of smooth forms.

3.3. Hodge decomposition theorem for compact Kähler manifolds.

$$H^k_{DR}(X) = \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(X).$$

For a proof, see page 116 in the book of Griffiths and Harris.

3.4. dd^c lemma for compact Kähler manifolds. Let T be a real smooth (p,p) form on X. If the cohomology class $\{T\}$ is zero in $H^{p,p}(X)$, then there exists a real smooth $(p-1,p-1)$ form ψ so that $T = dd^c\psi$. Similarly, if T is a real (p,p) current whose cohomology class is zero, then there is a real $(p-1,p-1)$ current R so that $T = dd^cR$. For a proof of this fact, see Lemma 8.6 page 311 in Demailly's book.

4. Meromorphic maps and dynamical degrees

Let X and Y be compact Kähler manifolds. Let $\pi_X, \pi_Y : X \times Y \to X, Y$ be the projections.

4.1. Meromorphic maps. Following Remmert, we define a meromorphic map $f : X \to Y$ to be an analytic variety $\Gamma_f \subset X \times Y$ (the graph of f) which is mapped properly onto X by the projection π_X such that outside a proper analytic subset $Z \subset X$, this map is biholomorphic. Moreover $\pi_X^{-1}(Z)$ has to be nowhere dense in Γ_f. Note that there is an (Zariski) open dense set $U \subset X$ for which $f|_U : U \to Y$ is a true holomorphic map, and the closure of the graph of $f|_U$ is exactly Γ_f.

The map f is dominant if $\pi_Y(\Gamma_f) = Y$. With the same set U in the previous paragraph, this means that $f|_U(U)$ is dense in Y.

Composition of two dominant meromorphic maps: Let $f : X \to Y$ and $g : Y \to Z$ be two dominant meromorphic maps. Then we can define their composition $h = g \circ f$ to be a dominant meromorophic map from X to Z in the following way: there are open dense sets $U \subset X$ and $V \subset Y$ so that $f|_U$ and $g|_V$ are holomorphic maps, and $f_U(U)$ is dense in Y. Then $f|_U^{-1}(V)$ is dense in X, and $g|_V \circ f|_{f|_U^{-1}(V)}$ is a holomorphic map from X onto a dense subset in Z. The closure of the graph of this map is an analytic variety Γ_h which defines a dominant meromorphic map $h : X \to Z$.

Define $\tau : X \times Y \to Y \times X$ be the map $\tau(x, y) = (y, x)$. The map f is bimeromorphic if the variety $\tau(\Gamma_f)$ of $Y \times X$ also defines a meromorphic map from $Y \to X$. We denote the map corresponding with $\tau(\Gamma_f)$ by f^{-1}. This notation is justified by the fact that $f \circ f^{-1} = Id_Y$ and $f^{-1} \circ f = Id_X$, here the compositions are in the sense of meromorphic maps as defined in the previous paragraph.

4.2. Pullback of smooth forms and of cohomology classes. Let $f : X \to Y$ be a dominant meromorphic map of compact Kähler manifolds. Let Γ_f be the graph of the map f.

Let α be a smooth (p, q) form on Y. We define a pullback $f^*(\alpha)$, which is a (p, q) current on X, by the following formula: $f^*(\alpha) = (\pi_X)_*([\Gamma_f] \wedge \pi_Y^*(\alpha))$.

It can be checked that if α is a closed or exact form then $f^*(\alpha)$ is a closed or exact current. Therefore the pullback on forms induces the pullback on cohomology groups $f^* : H^{p,q}(Y) \to H^{p,q}(X)$.

Similarly, the pushforward of a smooth form α is given by the formula $f_*(\alpha) = (\pi_Y)_*([\Gamma_f] \wedge \pi_X^*(\alpha))$.

4.3. Dynamical degrees. Let $f : X \to X$ be a dominant meromorphic selfmap of a compact Kähler manifold X, of dimension k. Let $f^n = f \circ f \ldots \circ f$ be the n-th iterate of f. We define dynamical degrees of f by one of the following two equivalent ways:

Way 1: Let ω_X be a Kähler $(1,1)$ form on X, and let $0 \leq p \leq k$. Then the p-th dynamical degree of f is given by

$$\delta_p(f) = \lim_{n\to\infty} \Big(\int_X (f^n)^* \omega_X^p \wedge \omega_X^{k-p}\Big)^{1/n}.$$

Way 2: Let $r_p(f^n)$ be the spectral radius of the linear map $(f^n)^* : H^{p,p}(X) \to H^{p,p}(X)$. Then

$$\delta_p(f) = \lim_{n\to\infty} (r_p(f^n))^{1/n}.$$

The following result belongs to Dinh and Sibony [**38**][**39**]:

THEOREM 2.3. *The dynamical degrees are bimeromorphic invariants. This means that if $\pi : X \to Y$ is a bimeromorphic map, $f : X \to X$ is a dominant meromophic map and $g = \pi \circ f \circ \pi^{-1} : Y \to Y$, then $\delta_p(f) = \delta_p(g)$ for all $0 \leq p \leq k$.*

Some other properties of dynamical degrees:

If f is holomorphic then $(f^n)^* = (f^*)^n : H^{p,p}(X) \to H^{p,p}(X)$, and hence $\delta_p(f) = r_p(f)$. For general meromorphic map, the formula is not true.

$\delta_0(f) = 1$ and $\delta_k(f)$ =the topological degree of f, i.e. the number of inverse images by the map f of a generic point in X.

The function $p \to \delta_p(f)$ is log-concave. In particular, $\delta_p(f)^2 \geq \delta_{p-1}(f)\delta_{p+1}(f)$ for all $1 \leq p \leq k-1$.

When $X = \mathbb{P}^k$, then a meromorphic selfmap f of X is a rational map (Chow's theorem). Such a map f can be written as $f(z) = [P_0(z_0, \dots, z_k) : P_1(z_0, \dots, z_k) : \dots : P_k(z_0, \dots, z_k)]$ in homogeneous coordinates $z = [z_0 : z_1 : \dots : z_k]$. Here the functions $P_j(z_0, \dots, z_k)$ are homogeneous polynomials of the same degree d, and the largest common divisor of polynomials $P_0, \dots, P_k$ is 1. Then we call d the degree of f, and denote $deg(f) = d$. In this case, $H^{1,1}(\mathbb{P}^k)$ has dimension 1, and $f^* : H^{1,1}(X) \to H^{1,1}(X)$ is the multiplication by $deg(f)$. Therefore $\delta_1(f) = \lim_{n\to\infty} deg(f^n)^{1/n}$, and hence $\delta_1(f)$ is also called the degree growth of f.

4.4. Entropy. Let $f : X \to X$ be a surjective holomorphic map. Let d be a metric on X. A subset E of X is called (n, ϵ)-separated if for any pair $x, y \in E$ then $\max\{d(f^i(x), f^i(y)) :\ 0 \leq i \leq n-1\} \geq \epsilon$. Denote by $N(n, \epsilon)$ the maximal cardinality of an (n, ϵ)-separated set. Then the topological entropy of f is given by

$$h_{top}(f) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n, \epsilon).$$

Gromov [**55**] and Yomdin [**71**] proved the following result, which relates the topological entropy of a holomorphic map to its dynamical degrees:

THEOREM 2.4. *If $f : X \to X$ is a surjective holomorphic map then*

$$h_{top}(f) = \max_{1\leq p\leq k} \log \delta_p(f).$$

If f is a meromorphic map, we can define in a similar fashion its topological entropy, replacing X by $\Omega_f = X - \bigcup_{n\in\mathbb{X}} f^n(\mathcal{I}_f)$, and replacing f by $f|_{\Omega_f}$ (see e.g. Guedj's survey paper [**56**]). Here Ω_f is not compact, and the metric on Ω_f is induced from that of X. However, the topological entropy such defined is not a bimeromorphic invariant (see the example below). Dinh and Sibony (see the cited papers) generalized Gromov-Yomdin's theorem as follows:

THEOREM 2.5. *If $f : X \to X$ is a dominant meromorphic map, then*

$$h_{top}(f) \leq \max_{1 \leq p \leq k} \log \delta_p(f).$$

The reverse inequality is not true in general: There are examples of dominant meromorphic maps f for which $h_{top}(f) = 0$ but $\max_{1 \leq p \leq k} \log \delta_p(f) > 0$. For example, Guedj [**58**] gave the following example: $f : \mathbb{P}^2 \to \mathbb{P}^2$ with $f[z : w : t] = [z^2 : wt + t^2 : t^2]$. Its first dynamical degree is 2, but its topological entropy is 0. This map is birationally equivalent to the following map $g : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ with $g([z : 1], [w : 1]) = ([z^2 : 1], [w + 1 : 1])$. This map g is holomorphic on $\mathbb{P}^1 \times \mathbb{P}^1$, and its first and second dynamical degrees are both 2, hence by Gromov-Yomdin's theorem its topological entropy is $\log 2$.

CHAPTER 3

Pullback of currents by meromorphic maps

1. The pullback operator

1.1. Introduction. Let X and Y be two compact Kähler manifolds, and let $f : X \to Y$ be a dominant meromorphic map. For a (p,p)-current T on Y, we seek to define a pullback $f^*(T)$ which has good properties. We let $\pi_X, \pi_Y : X \times Y \to X, Y$ be the two projections (When $X = Y$ we denote these maps by π_1 and π_2). Let $\Gamma_f \subset X \times Y$ be the graph of f, and let $\mathcal{C}_f \subset \Gamma_f$ be the "exceptional" set of $\pi_Y|\Gamma_f$, i.e. the smallest analytic subvariety of Γ_f so that the restriction of π_Y to $\Gamma_f - \mathcal{C}_f$ has fibers of dimension $dim(X) - dim(Y)$. For a set $B \subset Y$, we define $f^{-1}(B) = \pi_X(\pi_Y^{-1}(B) \cap \Gamma_f)$, and for a set $A \subset X$ we define $f(A) = \pi_Y(\pi_X^{-1}(A) \cap \Gamma_f)$.

If T is a smooth form on Y, then it is standard to define $f^*(T)$ as a current on X by the formula $f^*(T) = (\pi_X)_*(\pi_Y^*(T) \wedge [\Gamma_f])$, which descends to cohomology classes (see the preliminary results chapter). These considerations apply equally to continuous forms.

Our idea for pulling back a general (p,p) current T is as follows. Assume that we have a well-defined pullback $f^*(T)$. Then for any smooth form of complement bidegree α we should have

$$\int_X f^*(T) \wedge \alpha = \int_Y T \wedge f_*(\alpha).$$

Here the pushforward of the smooth form α is defined, similarly to the pullback defined above, as $f_*(\alpha) = (\pi_Y)_*(\pi_X^*(\alpha) \wedge [\Gamma_f])$. The wedge product in the integral of the RHS is not well-defined in general. To define it we use smooth approximations of either T or $f_*(\alpha)$. Fortunately, since Y is compact, any current T is of a finite order s. Moreover since $f_*(\alpha)$ is a DSH current (see Section in Chapter 1), we can use the regularization theorem in [**44**] to produce approximation by C^s forms $\mathcal{K}_n(f_*(\alpha))$ with desired properties. Then we define

$$\int_X f^*(T)\wedge\alpha = \lim_{n\to\infty}\int_Y T\wedge\mathcal{K}_n(f_*(\alpha)),$$

if the limit exists and is the same for such good approximations.

Details of the definition will be given in the next subsections. We will also discuss a related result on strict transform of quasi-pluri-subharmonic currents by meromorphic maps in Theorem 3.9. We end this subsection by stating some applications of the definition of pullback of currents.

-If $\pi_1(\mathcal{C}_f)$ has codimension $\geq p$ then we can pullback any positive closed (p,p) current by f.

-If f is p-analytic stable (see definition in Section 4) then we can construct invariant (p,p) currents for f.

-If T is a positive closed (p,p) current and f is a dominant meromorphic map so that $f^*(T)$ is well-defined, then $f^*(T)$ may no longer be positive.

The rest of this chapter is organized as follows. In the remaining of this Section we give definitions of good approximation schemes and of the pullback operator. In Sections 2,3, and 4, we state and prove various properties and applications of the definition. In the last Section we explore a simple but interesting quasi-automorphism in 3-dimension (the map J_X) and state some open questions.

1.2. Good approximation schemes. In this section we define good approximation schemes for DSH currents, which will be applied in particular to currents of the form $f_*(\alpha)$ where α is a smooth form. For the notions of positive currents and DSH currents, please see the preliminary chapter.

DEFINITION 3.1. *Let Y be a compact Kähler manifold. Let $s \geq 0$ be an integer. We define a good approximation scheme by C^s forms for DSH currents Y to be an assignment that for a DSH current T gives two sequences $\mathcal{K}_n^{\pm}(T)$ (here $n = 1, 2, \ldots$) where $\mathcal{K}_n^{\pm}(T)$ are C^s forms of the same bidegrees as T, so that $\mathcal{K}_n(T) = \mathcal{K}_n^{+}(T) - \mathcal{K}_n^{-}(T)$ weakly converges to T, and moreover the following properties are satisfied:*

1) Boundedness: If T is DSH then the DSH norms of $\mathcal{K}_n^{\pm}(T)$ are uniformly bounded.

2) Positivity: If T is positive then $\mathcal{K}_n^{\pm}(T)$ are positive.

3) Closedness: If T is closed then $\mathcal{K}_n^{\pm}(T)$ are closed.

4) Continuity: If T is DSH and $U \subset Y$ is an open set so that $T|_U$ is a continuous form then $\mathcal{K}_n^{\pm}(T)$ converges locally uniformly on U.

5) Linearity: If T_1 and T_2 are two DSH currents, then $\mathcal{K}_n^{\pm}(T_1 + T_2) = \mathcal{K}_n^{\pm}(T_1) + \mathcal{K}_n^{\pm}(T_2)$.

6) Self-adjointness: If T and S are DSH currents with complement bidegrees then

$$\lim_{n\to\infty}\Big[\int_Y \mathcal{K}_n(T) \wedge S - \int_Y T \wedge \mathcal{K}_n(S)\Big] = 0.$$

7) Compatibility with the differentials: $dd^c\mathcal{K}_n^{\pm}(T) = \mathcal{K}_n^{\pm}(dd^cT)$.

8) Convergence of supports: If A is compact and U is an open neighborhood of A, then there is $n_0 = n_0(U, A)$ such that if the support of T is contained in A and $n \geq n_0$ then $supp(\mathcal{K}_n(T))$ is contained in U.

Now we give examples of good approximation schemes. First, we recall the construction of the weak approximation for the diagonal K_n from Section 3 in [**39**]. Let $k = dim(Y)$. Let $\pi : \widetilde{Y \times Y} \rightarrow Y \times Y$ be the blowup of $Y \times Y$ at Δ_Y. Let $\widetilde{\Delta}_Y = \pi^{-1}(\Delta_Y)$ be the exceptional divisor. Then there is a closed smooth $(1,1)$ form γ and a negative quasi-plurisubharmonic function φ so that $dd^c\varphi = [\widetilde{\Delta}_Y] - \gamma$. We choose a strictly positive closed smooth $(k-1, k-1)$ form η so that $\pi_*([\widetilde{\Delta}_Y] \wedge \eta) = [\Delta_Y]$.

Observe that φ is smooth out of $[\widetilde{\Delta_Y}]$, and $\varphi^{-1}(-\infty) = \widetilde{\Delta_Y}$. Let $\chi : \mathbb{R} \cup \{-\infty\} \to \mathbb{R}$ be a smooth increasing convex function such that $\chi(x) = 0$ on $[-\infty, -1]$, $\chi(x) = x$ on $[1, +\infty]$, and $0 \leq \chi' \leq 1$. Define $\chi_n(x) = \chi(x+n) - n$, and $\varphi_n = \chi_n \circ \varphi$. The functions φ_n are smooth decreasing to φ, and $dd^c\varphi_n \geq -\Theta$ for every n, where Θ is a strictly positive closed smooth $(1,1)$ form so that $\Theta - \gamma$ is strictly positive. Then we define $\Theta_n^+ = dd^c\varphi_n + \Theta$ and $\Theta_n^- = \Theta^- = \Theta - \gamma$. Finally $K_n^{\pm} = \pi_*(\Theta_n^{\pm} \wedge \eta)$, and $K_n = K_n^+ - K_n^-$. By replacing $K_n^{\pm}(y_1, y_2)$ by $[K_n^{\pm}(y_1, y_2) + K_n^{\pm}(y_2, y_1)]/2$ if needed, we may assume that $K_n^{\pm}$ and K_n are symmetric, i.e. $K_n^{\pm}(y_1, y_2) = K_n^{\pm}(y_2, y_1)$.

Remark 3.1. *For the sake of simplicity, for a DSH current T on Y, and for the currents K_n as above, we will use the notation: $\widehat{K}_n(T) = (\pi_1)_*(\pi_2^*(T) \wedge K_n)$. We also use the similar notations $\widehat{K}_n^{\pm}(T)$. Let $\eta : Y \times Y \to Y \times Y$ be the automorphism $\eta(y_1, y_2) = (y_2, y_1)$. Then, since $K_n^{\pm}$ are symmetric, if θ is a smooth form on Y we have*

$$\begin{aligned}
\int_Y \theta \wedge \widehat{K}_n^{\pm}(T) &= \int_Y \theta \wedge (\pi_1)_*(\pi_2^*(T) \wedge K_n^{\pm}) = \int_{Y\times Y} \pi_1^*(\theta) \wedge \pi_2^*(T) \wedge K_n^{\pm} \\
&= \int_{Y\times Y} \eta^*(\pi_1^*(\theta) \wedge \pi_2^*(T) \wedge K_n^{\pm}) = \int_{Y\times Y} \pi_2^*(\theta) \wedge \pi_1^*(T) \wedge K_n^{\pm} \\
&= \int_Y \theta \wedge (\pi_2)_*(\pi_1^*(T) \wedge K_n^{\pm}).
\end{aligned}$$

(To show the equality between the furthest LHS and the RHS of this expression, we need only to do so for T is a smooth form, because both are continuous in the DSH topology. In the case when T is a smooth form, we used that $\eta^*(\pi_1^*(\theta) \wedge \pi_2^*(T) \wedge K_n^{\pm})$ $= \eta^*(\pi_1^*(\theta)) \wedge \eta^*(\pi_2^*(T) \wedge K_n^{\pm}) = \eta^*(\pi_1^*(\theta)) \wedge \eta^*(\pi_2^*(T) \wedge \eta^*(K_n^{\pm}) = \pi_2^*(\theta)) \wedge \pi_1^*(T) \wedge$ $K_n^{\pm}$.) Therefore $\widehat{K}_n^{\pm}(T)$ is also equal to $(\pi_2)_*(\pi_1^*(T) \wedge K_n^{\pm})$. Similarly, $\widehat{K}_n(T) =$ $(\pi_2)_*(\pi_1^*(T) \wedge K_n)$.

Let l be a large integer dependent on s, and let $(m_1)_n, \ldots, (m_l)_n$ be sequences of positive integers satisfying $(m_i)_n = (m_{l+1-i})_n$ and $\lim_{n\to\infty}(m_i)_n = \infty$ for any

$1\leq i\leq l$. We claim that if we choose $\mathcal{K}_n=\widehat{K}_{(m_1)_n}\circ\widehat{K}_{(m_2)_n}\circ\ldots\circ\widehat{K}_{(m_l)_n}$ then it satisfies conditions 1)-8), and thus give examples of good approximation schemes by C^s forms. Properties 1), 2), 3), 5), 7) and 8) follows immediately from the properties of the kernels $\widehat{K}_n$ (see Sections 3 and 4 in [**39**]).

THEOREM 3.1. *i) If T_1 is a $DSH^p(Y)$ current and T_2 is a continuous $(dim(Y)-p,dim(Y)-p)$ form on Y then*

$$\int_Y\widehat{K}_n^{\pm}(T_1)\wedge T_2=\int_Y T_1\wedge\widehat{K}_n^{\pm}(T_2).$$

ii) Let T be a DSH current. Then $\mathcal{K}_n(T)$ converges in $DSH^p(Y)$ to T.

PROOF. i) By Theorems 1.1 and 4.4 in [**39**], the LHS of the equality we want to prove is continuous for the DSH convergence w.r.t. T_1. The RHS of the equality is also continuous for the DSH convergence w.r.t. T_1. Hence using the approximation theorem for DSH currents of Dinh and Sibony, it suffices to prove the equality when T_1 is a smooth form, in which case it is easy to be verified as follows

$$\begin{aligned}\int_Y\widehat{K}_n^{\pm}(T_1)\wedge T_2 &= \int_Y(\pi_1)_*(\pi_2^*(T_1)\wedge K_n^{\pm})\wedge T_2=\int_{Y\times Y}\pi_2^*(T_1)\wedge K_n^{\pm}\wedge(\pi_1)^*(T_2)\\ &= \int_Y T_1\wedge(\pi_2)_*((\pi_1)_*(T_2)\wedge K_n^{\pm})=\int_Y T_1\wedge\widehat{K}_n^{\pm}(T_2).\end{aligned}$$

ii) Note that since $||\mathcal{K}_n(T)||_{DSH}\leq A||T||_{DSH}$, to prove ii) it suffices to show that $\mathcal{K}_n(T)$ converges weakly to T in the sense of currents.

We prove by induction on l. If $l=1$, ii) is the approximation theorem of Dinh and Sibony. To illustrate the idea of the proof, we show for example how to prove ii) for the case $l=2$ when knowing ii) for $l=1$. Hence we need to show that: For a smooth $(dim(Y)-p,dim(Y)-p)$ form α

$$\lim_{n_1,n_2\rightarrow\infty}\int_Y\widehat{K}_{n_2}\circ\widehat{K}_{n_1}(T)\wedge\alpha=\int_Y T\wedge\alpha.$$

Since α is smooth, by i) we have

$$\lim_{n_1,n_2\to\infty}\int_Y \widehat{K}_{n_2}\circ\widehat{K}_{n_1}(T)\wedge\alpha=\lim_{n_1,n_2\to\infty}\int_Y \widehat{K}_{n_1}(T)\wedge\widehat{K}_{n_2}(\alpha).$$

By the case $l=1$ we know that $\widehat{K}_{n_1}(T)$ converges to T in DSH^p. By properties of the kernels K_n, $\widehat{K}_{n_2}(\alpha)$ converges uniformly to α. Hence $\alpha-\widehat{K}_{n_2}(\alpha)$ is bound by $\epsilon_{n_2}\omega_Y^{dim(Y)-p}$, where $\epsilon_{n_2}\to 0$ as $n_2\to\infty$. Thus

$$|\int_Y \widehat{K}_{n_1}(T)\wedge\widehat{K}_{n_2}(\alpha)-\int_Y \widehat{K}_{n_1}(T)\wedge\alpha|\leq A\epsilon_{n_2},$$

where $A>0$ is independent of n_1 and n_2, and $\epsilon_{n_2}\to 0$ when $n_2\to\infty$. Letting limit when n_1,n_2 converges to ∞ and using the induction assumption for $l=1$, we obtain the claim for $l=2$. □

Property 6) follows from Theorem 3.1. In fact, by the results in [**39**], there is a number l_0 so that for $l'\geq l_0$ and T a DSH current, then $\widehat{K}_{m_1}\circ\ldots\circ\widehat{K}_{m_{l'}}(T)$ is a continuous form for any sequence $m_1,\ldots,m_{l'}$. We choose $l\geq 2l_0$ in the definition of $\mathcal{K}_n$. Given DSH currents T and S. Since $\mathcal{K}_n(T)=\widehat{K}_{(m_1)_n}(T')$ where $T'=\widehat{K}_{(m_2)_n}\circ\widehat{K}_{(m_l)_n}(T)$ is a continuous form, we have by Theorem 3.1

$$\int_Y \mathcal{K}_n(T)\wedge S=\int_Y \widehat{K}_{(m_2)_n}\circ\widehat{K}_{(m_l)_n}(T)\wedge\widehat{K}_{(m_1)_n}(S).$$

We can iterate this, note that we chose $l\geq l'$, to obtain

$$\int_Y \mathcal{K}_n(T)\wedge S=\int_Y T\wedge\widehat{K}_{(m_l)_n}\circ\widehat{K}_{(m_{l-1})_n}\circ\ldots\circ\widehat{K}_{(m_1)_n}(S).$$

By the property $(m_i)_n=(m_{l+1-i})_n$, we have $\widehat{K}_{(m_l)_n}\circ\widehat{K}_{(m_{l-1})_n}\circ\ldots\circ\widehat{K}_{(m_1)_n}(S)=\mathcal{K}_n(S)$. Therefore

$$\int_Y \mathcal{K}_n(T)\wedge S=\int_Y T\wedge\mathcal{K}_n(S),$$

and Property 6) holds for individual terms.

Property 4) is proved in the following result

PROPOSITION 3.1. *i) Let T_n be a sequence of $DSH^p(Y)$ currents converging in DSH to T. Assume that there is an open set $U \subset Y$ so that $T_n|_U$ are continuous forms, and T_n converges locally uniformly on U to T. Then $\widehat{K}_n^{\pm}(T_n)|_U$ are continuous and converges locally uniformly on U.*

ii) Let T be a $DSH^p(Y)$ current. Assume that there is an open set $U \subset Y$ so that $T|_U$ is a continuous form. Then for any positive integer l, $\mathcal{K}_n^{\pm}(T)|_U$ are continuous forms, and converges locally uniformly on U.

PROOF. i) Let $U_1 \subset\subset U_2 \subset\subset U_3 \subset\subset U$ be a relative compact open sets in U. We will show that $\widehat{K}_n^{\pm}(T_n)$ converges uniformly on U_1. Let $\chi_2 : Y \to [0,1]$ be a cutoff function for U_2 so that χ_2 is smooth, $\chi_2 = 1$ on U_2 and $\chi_2 = 0$ outside of U_3. We write $\widehat{K}_n^{\pm}(T_n) = \widehat{K}_n^{\pm}(\chi_2 T_n) + \widehat{K}_n^{\pm}((1-\chi_2)T_n)$. By assumptions, $\chi_2 T_n$ converges uniformly on Y to $\chi_2 T$, so there are ϵ_n decreasing to 0 as $n \to 0$ so that $-\epsilon_n \omega_Y^p \leq \chi_2 T_n - \chi_2 T \leq \epsilon_n \omega_Y^p$. Then

$$-\epsilon_n \widehat{K}_n^{\pm}(\omega_Y^p) \leq \widehat{K}_n^{\pm}(\chi_2 T_n) - \widehat{K}_n^{\pm}(\chi_2 T) \leq \epsilon_n \widehat{K}_n^{\pm}(\omega_Y^p).$$

Now $\widehat{K}_n^{-}(\omega_Y^p) = \widehat{K}^{-}(\omega_Y^p)$ is a smooth form, and hence $\widehat{K}_n^{+}(\omega_Y^p) = \widehat{K}_n(\omega_Y^p) - \widehat{K}^{-}(\omega_Y^p)$ is a sequence of smooth forms converging uniformly on Y, by applying Proposition 4.6 in [**39**] to ω_Y^p. Hence to prove i), it remains to show that $\widehat{K}_n^{\pm}((1-\chi_2)T_n)$ converges uniformly on U_1.

We let $\chi_1 : Y \to [0,1]$ be a cutoff function for U_1 so that χ_1 is smooth, $\chi_1 = 1$ on U_1 and $\chi_1 = 0$ outside of U_2. Then it suffices to show that $\chi_1 \widehat{K}_n^{\pm}((1-\chi_2)T_n)$ uniformly converges on Z. By definition, we have

$$\begin{aligned}\chi_1 \widehat{K}_n^{\pm}((1-\chi_2)T_n)(x) &= \int_Y \chi_1(x) K_n^{\pm}(x,y) \wedge (1-\chi_2(y)) T_n(y) dy \\ &= \int_Y \chi_1(x)(1-\chi_2(y)) K_n^{\pm}(x,y) \wedge T_n(y) dy.\end{aligned}$$

By definition of χ_1 and χ_2, the support of $\chi_1(x)(1-\chi_2(y))K_n^{\pm}(x,y)$ is contained in a fixed compact set of $Y\times Y-\Delta_Y$. Hence by definition of $K_n^{\pm}$, there is an n_0 and smooth forms $k^{\pm}(x,y)$ on $Y\times Y$ so that $\chi_1(x)(1-\chi_2(y))K_n^{\pm}(x,y)=k^{\pm}(x,y)$ for all $n\geq n_0$. Then for $n\geq n_0$ we have

$$\chi_1\widehat{K}_n^{\pm}((1-\chi_2)T_n)(x)=\int_Y k^{\pm}(x,y)\wedge T_n(y)dy,$$

and the RHS converges uniformly to $\int_Y k^{\pm}(x,y)\wedge T(y)dy$ since $T_n\rightharpoonup T$.

ii) We prove the claim for example for the case $l=1$ and $l=2$.

First, consider the case $l=1$. Then ii) follows by applying i) to the constant sequence $T_n=T$.

Now we consider the case $l=2$. Then $\mathcal{K}^+_{n_1,n_2}(T)=\widehat{K}^+_{n_2}\circ\widehat{K}^+_{n_1}(T)+\widehat{K}^-_{n_2}\circ\widehat{K}^-_{n_1}(T)$, and $\mathcal{K}^-_{n_1,n_2}(T)=\widehat{K}^+_{n_2}\circ\widehat{K}^-_{n_1}(T)+\widehat{K}^-_{n_2}\circ\widehat{K}^+_{n_1}(T)$. We show for example that $\widehat{K}^+_{n_2}\circ\widehat{K}^+_{n_1}(T)$ converges uniformly locally on U as both n_1 and n_2 go to ∞. We apply i) to the sequence $T_n=\widehat{K}^+_n(T)$. The two conditions of i) are not hard to check: First, by the case $l=1$ the sequence T_n converges locally uniformly on U. Second, by Theorem 3.1, $T_n=\widehat{K}_n(T)+\widehat{K}^-(T)\rightharpoonup T+\widehat{K}^-(T)$. □

1.3. Definition of the pullback operator. Now we give details of the definition of the pullback operator. Before giving the definition, let us state the following observation concerning the approach using all possible approximations. (This is in fact a general fact of linear operators: if a linear operator is continuous at one point then it is continuous everywhere.)

LEMMA 3.1. *Assume that for a positive closed (p,p) current T_0 and for every sequence of positive closed smooth forms $T_n^{\pm}$ whose masses $||T_n^{\pm}||$ are uniformly bounded and $T_n^+-T_n^-\rightharpoonup T_0$, then $f^*(T_n^+-T_n^-)$ converges to the same limit. Then the same property holds for any positive closed (p,p) current T.*

Proof. In fact, let $T_n^+ - T_n^-$ and $S_n^+ - S_n^-$ be two sequences weakly converging to a positive closed (p,p) current T, where $T_n^{\pm}$ and $S_n^{\pm}$ are positive closed smooth (p,p) forms having uniformly bounded masses. Let $H_n^+ - H_n^-$ be a sequence weakly converging to T_0 where $H_n^{\pm}$ are positive closed smooth (p,p) forms having uniformly bounded masses. Then $(T_n^+ + S_n^- + H_n^+) - (T_n^- + S_n^+ + H_n^-)$ is a sequence weakly converges to T_0 with the same property. Hence $f^*(T_n^+ + S_n^- + H_n^+) - f^*(T_n^- + S_n^+ + H_n^-)$ and $f^*(H_n^+ - H_n^-)$ converges to the same limit by assumption, and therefore we must have $f^*(T_n^+ + S_n^-) - f^*(T_n^- + S_n^+)$ weakly converges to 0. Hence $f^*(T_n^+ - T_n^-)$ and $f^*(S_n^+ - S_n^-)$ converges to the same limit. $\square$

Now we state the definition of pullback of currents for a dominant meromorphic map $f : X \to Y$ between compact Kähler manifolds. Recall that since Y is a compact manifold, any current on Y is of finite order (see the preliminary chapter). Upto this point, we have written T for a DSH current to be smoothen. In the definition below, T is a general current which is not necessarily DSH, and the current to be smoothen is $f_*(\alpha)$.

Definition 3.2. *Let T be a (p,p) current of order s_0. We say that $f^*(T)$ is well-defined if there is a number $s \geq s_0$ and a current S on X so that*

$$\lim_{n\to\infty} \int_Y T \wedge \mathcal{K}_n(f_*(\alpha)) = \int_X S \wedge \alpha,$$

for any smooth form α on X and any good approximation scheme by C^{s+2} forms. Then we write $f^*(T) = S$.

Note that by the self-adjointness property 6) of good approximation schemes, if T is a DSH current then $f^*(T) = S$ is well-defined iff there is a number $s \geq 0$ such that for any good approximation scheme by C^{s+2} forms $\mathcal{K}_n$ then $\lim_{n\to\infty} f^*(\mathcal{K}_n(T)) = S$.

Some examples of currents that can be pulled back by any meromorphic map are: continuous forms, subvarieties whose preimages have big codimension, measures having no mass on $\pi_X(\mathcal{C}_f)$, positive dd^c-closed $(1,1)$ currents... See the next sections for details.

2. Good properties of the pullback operator

2.1. Some simple properties. The operator f^* in Definition 3.2 has the following properties:

Lemma 3.2. *i) If T is a continuous (p,p) form (not necessarily DSH) then $f^*(T)$ is well-defined and coincides with the standard definition $f^*(T) := (\pi_1)_*(\pi_2^*(T)\wedge[\Gamma_f])$.*

ii) f^* is closed under linear combinations: If $f^*(T_1)$ and $f^\sharp(T_2)$ are well-defined, then so is $f^*(a_1T_1+a_2T_2)$ for any complex numbers a_1 and a_2. Moreover $f^*(a_1T_1 + a_2T_2) = a_1f^*(T_1) + a_2f^*(T_2)$.

iii) If T is DSH and $f^*(T)$ is well-defined, then the support of $f^*(T)$ is contained in $f^{-1}(supp(T))$.

iv) If T is closed then $f^*(T)$ is also closed, and in cohomology $\{f^*(T)\} = f^*\{T\}$.

Proof. Let $\mathcal{K}_n = \mathcal{K}_n^+ - \mathcal{K}_n^-$ be a good approximation scheme by C^2 forms.

i) If T is a continuous form, then $\mathcal{K}_n^{\pm}(T)$ uniformly converges on Y. Hence there are continuous forms T^+, T^- and constants ϵ_n decreasing to 0, so that $T = T^+ - T^-$ and $-\epsilon_n\omega_Y^p \leq K_n^{\pm}(T) - T^{\pm} \leq \epsilon_n\omega_Y^p$. Then

$$-\epsilon_n f^*(\omega_Y^p) \leq f^*(\mathcal{K}_n^{\pm}(T)) - f^*(T^{\pm}) \leq \epsilon_n f^*(\omega_Y^p),$$

and thus $f^*(\mathcal{K}_n^{\pm}(T))$ weakly converges to $f^*(T^{\pm})$. Therefore, $f^*(\mathcal{K}_n^+(T) - \mathcal{K}_n^-(T))$ weakly converges to $f^*(T^+) - f^*(T^-) = f^*(T)$. This shows that $f^*(T)$ is well-defined and coincides with the usual definition.

ii) Follows easily from the definition.

iii) If T is DSH, the result follows from the definition and the fact that support of $\mathcal{K}_n(T)$ converges to support of T.

iv) First we show that if $T = T_1 + dd^cT_2$ is closed, where T_1 is a (p,p) current and T_2 is a $(p-1,p-1)$ current both of order 0, and $f^*(T)$ is well-defined, then $f^*(T)$ is closed.

From the assumption, it follows that T_1 is closed. To show that $f^*(T)$ is closed, it suffices to show that if α is a d-exact $(dim(X)-p, dim(X)-p)$ smooth form, then

$$\int_X f^*(T)\wedge\alpha = 0.$$

In fact, by definition

$$\int_X f^*(T)\wedge\alpha = \lim_{n\to\infty}\int_Y T_1\wedge\mathcal{K}_n(f_*(\alpha)) + T_2\wedge dd^c\mathcal{K}_n(f_*(\alpha)).$$

By the dd^c lemma, there is a smooth form β so that $\alpha = dd^c(\beta)$. Then by the compatibility with differentials of good approximation schemes, we have $\mathcal{K}_n(f_*(\alpha)) = \mathcal{K}_n(f_*(dd^c\beta)) = dd^c\mathcal{K}_n(f_*(\beta))$ is d-exact. Thus each of the two integrals in the RHS of the above equality is 0, independent of n. Hence the limit is 0 as well.

Now we show that $\{f^*(T)\} = f^*\{T\}$. Let θ be a smooth closed form so that $\{T\} = \{\theta\}$. Then there is a current R so that $T-\theta = dd^c(R)$. If α is a closed smooth form then

$$\begin{aligned}\int_X (f^*(T)-f^*(\theta))\wedge\alpha &= \lim_{n\to\infty}\int_Y (T-\theta)\wedge\mathcal{K}_n(f_*(\alpha))\\ &= \lim_{n\to\infty}\int_Y dd^c(R)\wedge\mathcal{K}_n(f_*(\alpha))\\ &= \lim_{n\to\infty}\int_Y R\wedge\mathcal{K}_n(f_*(dd^c\alpha)) = 0,\end{aligned}$$

since $dd^c(\alpha) = 0$. This shows that $\{f^*(T)\} = \{f^*(\theta)\}$, and the latter is $f^*\{T\}$ by definition.

□

For a smooth form, we can also define its pullback by using any desingularization of the graph of the map. We have an analogous result

THEOREM 3.2. *Let $\widetilde{\Gamma_f}$ be a desingularization of Γ_f, and let $\pi : \widetilde{\Gamma_f} \to X$ and $g : \widetilde{\Gamma_f} \to Y$ be the induced maps of $\pi_X|\Gamma_f$ and $\pi_Y|\Gamma_f$. Thus $\widetilde{\Gamma_f}$ is a compact Kähler manifold, π is a modification, and g is a surjective holomorphic map so that $f = g \circ \pi^{-1}$. Let T be a (p,p) current on Y. If $g^*(T)$ is well-defined, then $f^*(T)$ is also well-defined. Moreover $f^*(T) = \pi_*(g^*(T))$.*

PROOF. Assume that $g^*(T)$ is well-defined with respect to number s in Definition 3.2. Let α be a smooth form on X and $\mathcal{K}_n$ a good approximation scheme by C^{s+2} forms on Y. Then $f_*(\alpha) = g_*(\pi^*\alpha)$. Since $\pi^*(\alpha)$ is smooth on $\widetilde{\Gamma_f}$ and $g^*(T)$ is well-defined, we have

$$\begin{aligned}\lim_{n\to\infty}\int_Y T\wedge\mathcal{K}_n(f_*\alpha) &= \lim_{n\to\infty}\int_Y T\wedge\mathcal{K}_n(g_*\pi^*\alpha)\\ &= \int_{\widetilde{\Gamma_f}} g^*(T)\wedge\pi^*\alpha = \int_X \pi_*g^*(T)\wedge\alpha,\end{aligned}$$

as wanted. □

The following result is a restatement of a result of Dinh and Sibony (section 5 in [**43**]), concerning pullback of DSH currents outside the set $\mathcal{C}_f$. Note that the resulting current is defined on $X - \pi_X(\mathcal{C}_f)$ only, and may not have locally finite mass across $\mathcal{C}_f$.

THEOREM 3.3. *Let θ be a smooth function on $X\times Y$ so that supp$(\theta)\cap\Gamma_f \subset \Gamma_f - \mathcal{C}_f$. Then for any DSH^p current T on Y, $(\pi_X)_*(\theta[\Gamma_f]\wedge\pi_Y^*(T))$ is well-defined (see also* [**63**]*).*

PROOF. In this proof we use the value $s = 0$ in Definition 3.2. The proof is the same as the proof of Lemma 3.3 in [**43**] using the following observations:

i) Lemma 3.1 in [**43**] applies for C^2 forms T_n. Hence Lemma 3.3 in [**43**] applies to C^2 forms T_n.

ii) Let us choose two different good approximation schemes by C^2 forms $\mathcal{K}_n = \mathcal{K}_n^+ - \mathcal{K}_n^-$ and $H_n = H_n^+ - H_n^-$. Then the sequences $\mathcal{K}_n^+(T) + H_n^-(T)$ and $\mathcal{K}_n^-(T) + H_n^+(T)$ converges in DSH to a same positive current.

iii) Apply Lemma 3.3 in [**43**] to the sequences $\mathcal{K}_n^+(T) + H_n^-(T)$ and $\mathcal{K}_n^-(T) + H_n^+(T)$, we conclude that in $\Gamma_f - \mathcal{C}_f$, the sequences $f^*(\mathcal{K}_n^+(T)) + f^*(H_n^-(T))$ and $f^*(\mathcal{K}_n^-(T)) + f^*(H_n^+(T))$ converges to a same current. Thus we have that the sequences $f^*(\mathcal{K}_n^+(T) - \mathcal{K}_n^-(T))$ and $f^*(H_n^+(T) - H_n^-(T))$ converges in $\Gamma_f - \mathcal{C}_f$ to a same current. □

2.2. Auxiliary results.

Lemma 3.3. *Let T be a positive closed (p,p) current on Y. Then there is a closed smooth (p,p) form θ on Y so that $\{\theta\} = \{T\}$ in cohomology, and moreover*

$$-A||T||\omega_Y^p \leq \theta \leq A||T||\omega_Y^p.$$

Here $A > 0$ is independent of T.

Proof. Let $\pi_1, \pi_2 : Y \times Y \to Y$ be the two projections, and let Δ_Y be the diagonal of Y. Let Δ be a closed smooth form on $Y \times Y$ representing the cohomology class of $[\Delta_Y]$. If we define

$$\theta = (\pi_1)_*(\pi_2^*(T) \wedge \Delta),$$

it is a smooth (p,p) current on Y having the same cohomology class as T. Since Y is compact, there is a constant $A > 0$ so that $A(\pi_1^*\omega_Y + \pi_2^*\omega_Y)^{dim(Y)} \pm \Delta$ are strongly positive forms. Since T is a positive current, it follows that

$$\theta = (\pi_1)_*(\pi_2^*(T) \wedge \Delta) \leq A(\pi_1)_*((\pi_1^*\omega_Y + \pi_2^*\omega_Y)^{dim(Y)} \wedge \pi_2^*(T)) = A||T||\omega_Y^p.$$

Similarly, we have also $\theta \geq -A||T||\omega_Y^p$. □

Consider a positive closed (p,p) current T on a compact Kähler manifold Y. It is known that (see Dinh and Sibony[**40**], Bost, Gillet and Soule[**13**]) there is a DSH

(p-1,p-1) current S and a closed smooth form α so that $T = \alpha + dd^c S$. Here S is a difference of two negative currents. When $p = 1$ or Y is a projective space, then we can choose S to be negative. However in general we can not choose S to be negative (see [**13**]). The following weaker conclusion is sufficient for the purpose here

Lemma 3.4. *Let T be a positive closed (p,p) current on a compact Kähler manifold Y. Then there is a closed smooth (p,p) form α and a negative DSH $(p-1,p-1)$ current S so that*

$$T \leq \alpha + dd^c S.$$

Moreover, there is a constant $C > 0$ independent of T so that $||\alpha||_{L^\infty} \leq C||T||$ and $||S|| \leq C||T||$. If T is strongly positive then we can choose S to be strongly negative.

Here $||.||_{L^\infty}$ is the maximum norm of a continuous form and $||.||$ is the mass of a positive or negative current.

Proof. Notations are as in the paragraph in Section 2 when we defined the kernels K_n. Define $H = \pi_*(\varphi\eta)$. Then H is a negative $(k-1,k-1)$ current on $Y \times Y$.

We write $\gamma = \gamma^+ - \gamma^-$ for strictly positive closed smooth $(1,1)$ forms $\gamma^\pm$. If we define $\Phi^\pm = \pi_*(\gamma^\pm \wedge \eta)$ then $\Phi^\pm$ are positive closed (k,k) currents with L^1 coefficients. In fact (see [**39**]) $\Phi^\pm$ are smooth away from the diagonal Δ_Y, and the singularities of $\Phi^\pm(y_1,y_2)$ and their derivatives are bounded by $|y_1 - y_2|^{-(2k-2)}$ and $|y_1 - y_2|^{-(2k-1)}$. Moreover

$$dd^c H = \pi_*(dd^c\varphi \wedge \eta) = \pi_*([\widetilde{\Delta}_Y] \wedge \eta - (\gamma^+ - \gamma^-) \wedge \eta) = [\Delta_Y] - (\Phi^+ - \Phi^-).$$

Consider $S_1 = (\pi_1)_*(H \wedge \pi_2^*(T))$ and $R_1^\pm = (\pi_1)_*(\Phi^\pm \wedge T)$. Then S_1 is a negative current, and $R_1^\pm$ are positive closed currents. Moreover

$$dd^c S_1 = (\pi_1)_*(dd^c H \wedge \pi_2^*(T)) = T - R_1^+ + R_1^-.$$

Therefore $T \leq R_1^+ + dd^c S_1$. Moreover R_1^+ is a current with L^1 coefficients, and there is a constant $C_1 > 0$ independent of T so that $||S_1||, ||R_1^+||_{L^1} \leq C_1 ||T||$ (see e.g. Lemma 2.1 in [**39**]).

If we apply this process for R_1^+ instead of T we find a positive closed current R_2^+ with coefficients in $L^{1+1/(2k+2)}$ and a negative current S_2 so that $R_1^+ \leq R_2^+ + dd^c S_2$. Moreover

$$||R_2^+||_{L^{1+1/(2k+2)}}, ||S_2|| \leq C_2 ||R_1^+||_{L^1} \leq C_1 C_2 ||T||$$

for some constant $C_2 > 0$ independent of T. After iterating this process a finite number of times we find a continuous form R and a negative current S so that $T \leq R + dd^c S$. Moreover, $||R||_{L^\infty}, ||S|| \leq C||T||$ for some constant $C > 0$ independent of T. Since we can bound R by ω_Y^p upto a multiple constant of size $||R||_{L^\infty}$, we are done. $\square$

THEOREM 3.4. *Let T_j and T be (p,p) currents of order s_0. Assume that $-S_j \leq T - T_j \leq S_j$ for any j, where S_j are positive closed (p,p) currents with $||S_j|| \to 0$ as $j \to \infty$.*

1) If $f^*(T_j)$ is well-defined for any j with the same number s in Definition 3.2, then $f^*(T)$ is well-defined. Moreover $f^*(T_j)$ weakly converges to $f^*(T)$.

2) If $f^*(dd^c T_j)$ is well-defined for any j with the same number s in Definition 3.2, then $f^*(dd^c T)$ is well-defined. Moreover $f^*(dd^c T_j)$ weakly converges to $f^*(dd^c T)$.

Note that when $p = 0$, a closed $(0,0)$ current on X is a constant, hence the S_j in Theorem 3.4 are positive constants converging to zero.

PROOF. i) Let $\mathcal{K}_n = \mathcal{K}_n^+ - \mathcal{K}_n^-$ be a good approximation scheme by C^{s+2} forms. Let α be a strongly positive smooth $(k-p, k-p)$ form on X. then $f_*(\alpha)$ is a strongly positive form. Therefore $\mathcal{K}_n^\pm f_*(\alpha)$ are strongly positive forms of class C^2. Since

$-S_j \leq T_j - T \leq S_j$, we obtain

$$-\int_X S_j \wedge \mathcal{K}_n^{\pm} f_*(\alpha) \leq \int_X (T_j - T) \wedge \mathcal{K}_n^{\pm} f_*(\alpha) \leq \int_X S_j \wedge \mathcal{K}_n^{\pm} f_*(\alpha).$$

There is a constant $A > 0$ independent of α so that $A||\alpha||_{L^\infty}\omega_X^{k-p} \pm \alpha$ are strongly positive forms. Then $A||\alpha||_{L^\infty} f_*(\omega_X^{k-p}) \pm f_*(\alpha)$ are strongly positive forms on X. Hence we have

$$\int_X S_j \wedge \mathcal{K}_n^{\pm} f_*(\alpha) \leq A||\alpha||_{L^\infty} \int_X S_j \wedge \mathcal{K}_n^{\pm} f_*(\omega_X^{k-p}).$$

The latter integral can be computed cohomologously, hence can be bound as

$$\begin{aligned} A||\alpha||_{L^\infty} \int_X S_j \wedge \mathcal{K}_n^{\pm} f_*(\omega_X^{k-p}) &\leq A||\alpha||_{L^\infty}||S_j|| \times ||\mathcal{K}_n^{\pm} f_*(\omega_X^{k-p})|| \\ &\leq A||\alpha||_{L^\infty}||S_j|| \times ||f_*(\omega_X^{k-p})||. \end{aligned}$$

The latter inequality comes from the properties of good approximation schemes. Hence,

$$-A||\alpha||_{L^\infty}||S_j|| \leq \int_X (T_j - T) \wedge \mathcal{K}_n^{\pm} f_*(\alpha) \leq A||\alpha||_{L^\infty}||S_j||. \tag{2}$$

Since $f^*(T_j)$ are well-defined for all j, if we take limit as $n \to \infty$ in (2), we get

$$\begin{aligned} -A||\alpha||_{L^\infty}||S_j|| &\leq \int_X f^*(T_j) \wedge \alpha - \limsup_{n\to\infty} \int_X T \wedge \mathcal{K}_n f_*(\alpha) \\ &\leq \int_X f^*(T_j) \wedge \alpha - \liminf_{n\to\infty} \int_X T \wedge \mathcal{K}_n f_*(\alpha) \\ &\leq A||\alpha||_{L^\infty}||S_j||. \end{aligned}$$

Since $||S_j|| \to 0$, taking limit as $j \to \infty$ shows that

$$L(\alpha) := \lim_{n\to\infty} \int_X T \wedge \mathcal{K}_n f_*(\alpha)$$

exists, and moreover it satisfies

$$-A||\alpha||_{L^\infty}||S_j|| \leq \int_X f^*(T_j) \wedge \alpha - L(\alpha) \leq A||\alpha||_{L^\infty}||S_j||, \tag{3}$$

for all j, and all strongly positive smooth $(dim(X) - p, dim(X) - p)$ form α. Since any smooth $(dim(X) - p, dim(X) - p)$ form α is the difference of two strongly positive smooth $(dim(X)-p, dim(X)-p)$ forms α_1 and α_2 whose L^∞ norms are uniformly bounded (up to a multiplicative constant) by $||\alpha||_{L^\infty}$, it follows that (3) holds for any smooth form α. From this, it follows easily that the assignment $\alpha \mapsto L(\alpha)$ is a well-defined functional on smooth forms α. Now we show that it is a current on X. For this end, it suffices to show that if α_n are smooth forms so that $||\alpha_n||_{C^s} \rightarrow 0$ for any fixed $s \geq 0$ then $L(\alpha_n) \rightarrow 0$. This follows easily from (3) by first taking limit when $n \rightarrow \infty$ and then taking limit when $j \rightarrow \infty$, using the assumptions that $f^*(T_j)$ are currents, hence

$$\lim_{n\rightarrow\infty}\int_X f^*(T_j)\wedge\alpha_n = 0,$$

for any j.

ii) The proof is similar to the proof of i), with a small change: The estimate (2) is modified to

$$-A||dd^c\alpha||_{L^\infty}||S_j|| \leq \int_X (T_j - T)\wedge\mathcal{K}_n^{\pm}f_*(dd^c\alpha) \leq A||dd^c\alpha||_{L^\infty}||S_j||.$$

□

2.3. Main results. The first result applies for the case when $\pi_X(\mathcal{C}_f)$ has codimension $\geq p$, which generalizes a result proved by Dinh and Sibony in the case of projective spaces (see Proposition 5.2.4 in [**44**])

THEOREM 3.5. *Let X and Y be two compact Kähler manifolds. Let $f : X \rightarrow Y$ be a dominant meromorphic map. Assume that $\pi_X(\mathcal{C}_f)$ is of codimension $\geq p$. Then $f^*(T)$ is well-defined for any positive closed (p,p) current T on Y. Moreover the following continuity holds: if T_j are positive closed (p,p) currents weakly converging to T then $f^*(T_j)$ weakly converges to $f^*(T)$.*

PROOF. We follow the proof of Proposition 5.2.4 in [**44**] with some appropriate modifications. Let $\mathcal{K}_n = \mathcal{K}_n^+ - \mathcal{K}_n^-$ be a good approximation scheme by C^2 forms.

a) First we show that $f^*(T)$ is well-defined for any positive closed (p,p) current T.

Let θ be a smooth closed (p,p) form so that $\{\theta\} = \{T\}$ in cohomology classes. Since $T = (T-\theta)+\theta$, by Lemma 3.2, to show that $f^*(T)$ is well-defined, it is enough to show that $f^*(T-\theta)$ is well-defined. By dd^c lemma (see also [**40**]), there is a DSH current R so that $T-\theta = dd^c(R)$. Hence to show that $f^*(T-\theta)$ is well-defined, it is enough to show that $f^*(R)$ is well-defined.

We can write $\mathcal{K}_n(R) = R_{1,n} - R_{2,n}$, where $R_{i,n}$ are positive $(p-1,p-1)$ forms of class C^2, and $dd^c(R_{i,n}) = \Omega^+_{i,n} - \Omega^-_{i,n}$, where $\Omega^{\pm}_{i,n}$ are positive closed C^2 (p,p) forms. Moreover, $||R_{i,n}||$ and $||\Omega^{\pm}_{i,n}||$ are uniformly bounded.

i) First we show that $||f^*(R_{i,n})||$ are uniformly bounded. Theorem 3.3 implies that $f^*(R_{i,n})$ converges in $X - \pi_X(\mathcal{C}_f)$ to a current. Since the codimension of $\pi_X(\mathcal{C}_f)$ is $\geq p$, it is weakly p-pseudoconvex (see Lemma 5.2.2 in [**44**]). Hence there exists a smooth $(dim(X)-p, dim(X)-p)$ form Θ defined on X so that $dd^c\Theta \geq 2\omega_X^{dim(X)-p+1}$ on $\pi_X(\mathcal{C}_f)$. We can choose a small neighborhood V of $\pi_X(\mathcal{C}_f)$ so that $dd^c\Theta \geq \omega_X^{dim(X)-p+1}$ on V. Since $R_{i,n}$ is a positive C^2 form, $f^*(R_{i,n})$ is well defined and is a positive current. Since $f^*(R_{i,n})$ converges in $X - \pi_X(\mathcal{C}_f)$ to a current, it follows that $||f^*(R_{i,n})||_{X-V}$ is bounded. Because

$$||f^*(R_{i,n})||_X = ||f^*(R_{i,n})||_{X-V} + ||f^*(R_{i,n})||_V,$$

to show that $||f^*(R_{i,n})||_X$ is bounded, it is enough to estimate $||f^*(R_{i,n})||_V$. We have

$$\begin{aligned} ||f^*(R_{i,n})||_V &= \int_V f^*(R_{i,n}) \wedge \omega_X^{dim(X)-p+1} \leq \int_V f^*(R_{i,n}) \wedge dd^c(\Theta) \\ &= \int_X f^*(R_{i,n}) \wedge dd^c(\Theta) - \int_{X-V} f^*(R_{i,n}) \wedge dd^c(\Theta). \end{aligned}$$

The term

$$|\int_{X-V} f^*(R_{i,n}) \wedge dd^c(\Theta)|$$

can be bound by $||f^*(R_{i,n})||_{X-V}$, and thus is bounded. We estimate the other term: Since X is compact

$$\begin{aligned} |\int_X f^*(R_{i,n})\wedge dd^c(\Theta)| &= |\int_X dd^c f^*(R_{i,n})\wedge\Theta| = |\int_X f^*(dd^c R_{i,n})\wedge\Theta| \\ &= |\int_X f^*(\Omega_{i,n}^+ - \Omega_{i,n}^-)\wedge\Theta|. \end{aligned}$$

Since $\Omega_{i,n}^{\pm}$ are positive closed C^2 forms, $f^*(\Omega_{i,n}^{\pm})$ are well-defined and are positive closed currents. Choose a constant $A>0$ so that $A\omega_X^{dim(X)-p}\pm\Theta$ are strictly positive forms, we have

$$\begin{aligned} & |\int_X f^*(\Omega_{i,n}^+ - \Omega_{i,n}^-)\wedge\Theta| \\ \leq\ & |\int_X f^*(\Omega_{i,n}^+)\wedge\Theta| + |\int_X f^*(\Omega_{i,n}^-)\wedge\Theta| \\ \leq\ & A\int_X f^*(\Omega_{i,n}^+)\wedge\omega_X^{dim(X)-p} + A\int_X f^*(\Omega_{i,n}^-)\wedge\omega_X^{dim(X)-p}. \end{aligned}$$

Since $\Omega_{i,n}^{\pm}$ are positive closed currents with uniformly bounded norms, the last integrals are uniformly bounded as well.

ii) From i) we see that for any good approximation scheme by C^2 forms $\mathcal{K}_n$, the sequence $f^*(R_{1,n})-f^*(R_{2,n})$ has a convergent sequence. We now show that the limit is unique, hence complete the proof of Theorem 3.5. So let τ be the limit of the sequence $f^*(R_{1,n})-f^*(R_{2,n})$. Such a τ is a DSH^{p-1} current by the consideration in i). Let $H_n=H_n^+-H_n^-$ be another good approximation scheme by C^2 forms, and let τ' be the corresponding limit, which is in DSH^{p-1}. We want to show that $\tau=\tau'$. or equivalently, to show that $\tau-\tau'=0$.

By Theorem 3.3, $\tau-\tau'=0$ in $X-\pi_X(\mathcal{C}_f)$. Hence support of $\tau-\tau'$ is contained in $\pi_X(\mathcal{C}_f)$. Since $\tau-\tau'$ is in DSH^{p-1}, it is a $\mathbb{C}$-flat $(p-1,p-1)$ current (see the preliminary chapter or see Bassanelli [**3**]). Because the codimension of $\pi_X(\mathcal{C}_f)$ is $\geq p$, it follows by Federer-type support theorem for $\mathbb{C}$-flat currents (see the preliminary chapter or see Theorem 1.13 in [**3**]) that $\tau-\tau'=0$ identically.

b) Finally, we show that if T_j are positive closed (p,p) currents converging in DSH to T then $f^*(T_j)$ weakly converges to $f^*(T)$.

We let $\pi_1, \pi_2 : Y\times Y \rightarrow Y$ be the projections, and let Δ_Y be the diagonal. As in the proof of Lemma 3.3, we choose Δ to be a smooth closed $(dim(Y), dim(Y))$ on Y having the same cohomology class with $[\Delta_Y]$. We write $\Delta = \Delta^+ - \Delta^-$, where $\Delta^{\pm}$ are strongly positive smooth closed $(dim(Y), dim(Y))$ forms. If we define $\phi_j^{\pm} = (\pi_1)_*(\pi_2^*(T_j)\wedge\Delta^{\pm})$ and $\phi^{\pm} = (\pi_1)_*(\pi_2^*(T)\wedge\Delta^{\pm})$, then $\{T_j\} = \{\phi_j^+ - \phi_j^-\}$ and $\{T\} = \{\phi^+ - \phi^-\}$. Moreover, $\phi_j^{\pm}$ are positive closed smooth forms converging uniformly to $\phi^{\pm}$. Hence $f^*(\phi_j^{\pm})$ weakly converges to $f^*(\phi^{\pm})$. Thus to show that $f^*(T_j)$ weakly converges to $f^*(T)$, it is enough to show that $f^*(T_j - \phi_j)$ weakly converges to $f^*(T-\phi)$, where we define $\phi_j = \phi_j^+ - \phi_j^-$ and $\phi = \phi^+ - \phi^-$.

By Proposition 2.1 in [**40**], there are positive $(p-1,p-1)$ currents $R_j^{\pm}$ and $R^{\pm}$ so that $T_j - \phi_j = dd^c(R_j^+ - R_j^-)$, $T - \phi = dd^c(R^+ - R^-)$. Moreover, we can choose these in such a way that $R_j^{\pm}$ converges in DSH to $R^{\pm}$. From the proof of a), f^* is well-defined on the set of DSH^{p-1} currents. Thus to prove b) we need to show only that $f^*(R_j^{\pm})$ weakly converges to $f^*(R^{\pm})$.

By Theorem 3.3, on $X - \pi_X(\mathcal{C}_f)$ the currents $f^*(R_j^{\pm})$ and $f^*(R^{\pm})$ are the same as the currents $f^o(R_j^{\pm})$ and $f^o(R^{\pm})$ defined in [**43**]. Hence by the results in [**43**], it follows that $f^*(R_j^{\pm})$ weakly converges in $X - \pi_X(\mathcal{C}_f)$ to $f^*(R^{\pm})$. Thus as in the proof of a), to show that $f^*(R_j^{\pm})$ weakly converges to $f^*(R^{\pm})$, it suffices to show that $||f^*(R_j)||_{DSH}$ is uniformly bounded.

The current $f^*(R_j)$ is the limit of $f^*(\mathcal{K}_n(R_j))$. As in a), we write $K_n(R_j) = R_{j,n}^+ - R_{j,n}^-$ where $R_{j,n}^{\pm}$ are positive $DSH^{p-1}(Y)$ forms of class C^2. Moreover, by the properties of good approximation schemes, there is a constant $A > 0$ independent of j and n so that $||R_{j,n}^{\pm}||_{DSH} \leq A||R_j^{\pm}||_{DSH}$. It can be seen from the proof of a) that $f^*(R_j)$ is a DSH^{p-1} current. Moreover $||f^*(dd^cR_j)||_{DSH}$, which can be bound using intersections of cohomology classes, is $\leq A||R_j||_{DSH}$, where $A > 0$ is independent of j.

We choose an open neighborhood V of $\pi_X(\mathcal{C}_f)$ and a form Φ as in the proof of a). Then we can see from a) that

$$||f^*(R_j)||_{DSH} \leq A||f^*(R_j)||_{X-V,DSH} + A||f^*(dd^c R_j)||_{DSH},$$

where $A > 0$ is a constant independent of j, and $||f^*(R_j)||_{X-V,DSH}$ means the DSH norm of $f^*(R_j)$ computed on the set $X-V$. From the results in [**43**], $||f^*(R_j)||_{X-V,DSH}$ is uniformly bounded. The term $||f^*(dd^c R_j)||_{DSH}$ was shown above to be uniformly bounded as well. Thus $||f^*(R_j)||_{DSH}$ is uniformly bounded as desired. $\square$

Theorem 3.5 is a special case of the following result (choose $A = Y$ in Theorem 3.6).

Theorem 3.6. *Let X and Y be two compact Kähler manifolds. Let $f : X \rightarrow Y$ be a dominant meromorphic map. Let $A \subset Y$ be a closed subset so that $f^{-1}(A) \cap \pi_X(\mathcal{C}_f) \subset V$ where V is an analytic subvariety of X having codim $\geq p$. If T is a positive closed (p,p)-current on Y which is continuous on $Y - A$, then $f^*(T)$ is well-defined. Moreover, the following continuity holds: If $T_n^{\pm}$ are positive closed continuous (p,p) forms so that $||T_n^{\pm}||$ are uniformly bounded, $T_n^+ - T_n^- \rightharpoonup T$, and $T_n^{\pm}$ locally uniformly converges on $Y - A$, then $f^*(T_n^+ - T_n^-) \rightharpoonup f^*(T)$.*

Proof. Let θ be a closed smooth form on Y having the same cohomology class as T. Since T is continuous on $U = X - A$, there are DSH^{p-1} currents $R^{\pm}$ so that $T - \theta = dd^c(R^+) - dd^c(R^-)$, where $R^{\pm}|_U$ are continuous (see Proposition 2.1 in [**40**]). As in the proof of the Theorem 3.5, we will show that $f^*(R^{\pm})$ are well-defined. Since $f^{-1}(A) \cap \pi_X(\mathcal{C}_f) \subset V$, where V is of codimension $\geq p$, it is enough as before to show that $f^*(\mathcal{K}_n^{\pm}(R^{\pm}))$ have bounded masses outside a small neighborhood of $f^{-1}(A) \cap \pi_X(\mathcal{C}_f)$. First, by the proof of Theorem 3.5, $f^*(\mathcal{K}_n^{\pm}(R^{\pm}))$ have bounded masses outside a small neighborhood of $\pi_X(\mathcal{C}_f)$. Hence it remains to show that $f^*(\mathcal{K}_n^{\pm}(R^{\pm}))$ have bounded masses outside a small neighborhood of $f^{-1}(A)$.

Let B be a small neighborhood of $f^{-1}(A)$. Then there is a cutoff function χ for A, so that $f^{-1}(supp(\chi)) \subset B$. We write

$$f^*(\mathcal{K}_n^{\pm}(R^{\pm})) = f^*(\chi\mathcal{K}_n^{\pm}(R^{\pm})) + f^*((1-\chi)\mathcal{K}_n^{\pm}(R^{\pm})).$$

The first current has support in B, and hence has no contribution for the mass of $f^*(\mathcal{K}_n^{\pm}(R^{\pm}))$ outside B. By properties of good approximation schemes by C^2 forms, $(1-\chi)\mathcal{K}_n^{\pm}(R^{\pm})$ uniformly converges to a continuous form on Y, and hence $f^*((1-\chi)\mathcal{K}_n^{\pm}(R^{\pm}))$ has uniformly bounded masses on X, which is what wanted to prove.

To complete the proof, we need to show the continuity stated in the theorem. This continuity can be proved using the arguments from the first part of the proof, and from part b) of the proof of Theorem 3.5 and the proof of Proposition 3.1. □

Example 1: In [**5**], Bedford and Kim studied the linear fractional pseudo-automorphisms. These are birational selfmaps f of rational 3-manifolds X so that both f and f^{-1} have no exceptional hypersurfaces. Hence we can apply Theorem 3.5 to pullback and pushforward any positive closed $(2,2)$ current on X. The map J_X in Subsection 5.1 is also a pseudo-automorphism.

As another consequence, we have the following result on pulling back of varieties:

Corollary 3.1. *Let f, X, Y be as in Theorem 3.6. Let V be an analytic variety of Y of codim p. Assume that $f^{-1}(V)$ has codim $\geq p$. Then $f^*[V]$ is well-defined, and supported in $f^{-1}(V)$.*

Reamark that in Corollary 3.1, even if V is an effective variety $f^*[V]$ may be negative (see Corollary 3.7).

The assumptions in Corollary 3.1 are optimal, as can be seen from

Example 2: Let Y = a compact Kähler 3-fold, and let L_0 be an irreducible smooth curve in Y. Let $\pi : X \to Y$ be the blowup of Y along L_0. If L is an irreducible curve in Y which does not coincide with L_0 then $\pi^{-1}(L)$ has dimension 1, hence $\pi^*[L]$ is well-defined. In contrast, it is expected that $\pi^*[L_0]$ is not well-defined. One explanation (which is communicated to us by Professor Tien Cuong Dinh, see also the introduction in [**2**]) is that if $\pi^*[L_0]$ was to be defined, then it should be a special $(2,2)$ current on the hypersurface $\pi^{-1}(L_0)$. However, we have too many $(2,2)$ currents on that hypersurface to point out a special one.

We now turn to the pullback of a positive closed (p,p) current T in general. Let the Siu's decomposition of T (see the preliminary results chapter) be

$$T = R + \sum_{j=1}^{\infty} \lambda_j [V_j].$$

Theorem 3.7. *Notations are as above. Assume that for any irreducible variety V of codimension p contained in $E(T)$, then $f^{-1}(V)$ has codimension $\geq p$. Then $f^*(\sum_{j=1}^{\infty} \lambda_j [V_j])$ is well-defined and is equal to $\sum_{j=1}^{\infty} \lambda_j f^*[V_j]$. Hence $f^*(T)$ is well-defined iff $f^*(R)$ is well-defined.*

Proof. By assumption and Corollary 3.1, if V is an analytic variety of codimension p contained in $E(T)$, then $f^*[V]$ is well-defined with the number $s = 0$ in Definition 3.2. Hence the currents

$$W_N = \sum_{j=1}^{n} \lambda_j [V_j]$$

can be pulled back with the same number $s = 0$ in Definition 3.2, here N is a positive integer. Since $0 \leq \sum_j \lambda_j [V_j] - W_N = S_N$ where $S_N \rightharpoonup 0$ as $N \to \infty$, by Theorem 3.4 it follows that $f^*(\sum_j \lambda_j [V_j]) = \sum_j \lambda_j f^*[V_j]$ is well-defined. $\square$

3. Compatibility with previous definitions

We now compare our results with the results in previous papers.

The pullback of positive closed $(1,1)$ currents was defined by Meo [**63**] for finite holomorphic maps between complex manifolds (not necessarily compact or Kähler). Our definition coincides with his in the case of compact Kähler manifolds

COROLLARY 3.2. *Let X and Y be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Let T be a positive closed $(1,1)$-current on Y. Then $f^*(T)$ is well-defined, and coincides with the usual definition.*

PROOF. Since $\pi_X(\mathcal{C}_f)$ is a proper analytic subvariety of X, it has codimension ≥ 1, thus we can apply Theorem 3.5. □

More generally, we can pullback a function dominated by a quasi-PSH function φ (see [**43**] for pulling back of a form whose coefficients are bounded by a quasi-PSH function).

PROPOSITION 3.2. *Let ψ be a function bounded by a quasi-PSH function φ. Then the $(0,0)$ current defined by ψ can be pulled back by f. The resulting current $f^*(\psi)$ can be represented by an L^1 function.*

PROOF. Without loss of generality, we may assume that $0 \geq \psi \geq \varphi$. We will show the existence of a current S so that for any smooth form α and any good approximation scheme by C^2 forms $\mathcal{K}_n$ then

$$\lim_{n\to\infty}\int_Y \psi\wedge\mathcal{K}_n(f_*(\alpha)) = \int_X S\wedge\alpha. \tag{4}$$

We define linear functionals S_n and $S_n^{\pm}$ on top forms on X by the formulas

$$\begin{aligned} < S_n,\alpha > &= \int_Y \psi\wedge\mathcal{K}_n(f_*(\alpha)),\\ < S_n^{\pm},\alpha > &= \int_Y \psi\wedge\mathcal{K}_n^{\pm}(f_*(\alpha)). \end{aligned}$$

Then $S_n = S_n^+ - S_n^-$, and it can be checked that $S_n^{\pm}$ are negative $(0,0)$ currents, and hence S_n is a current of order 0. Moreover, if α is a positive smooth measure then

$$\begin{aligned} 0 \geq < S_n^{\pm}, \alpha > &= \int_Y \psi \wedge \mathcal{K}_n^{\pm}(f_*(\alpha)) \\ &\geq \int_Y \varphi \wedge \mathcal{K}_n^{\pm}(f^*(\alpha)) \\ &= \int_X f^*(\mathcal{K}_n^{\pm}(\varphi)) \wedge \alpha. \end{aligned}$$

Thus $0 \geq S_n^{\pm} \geq f^*(\mathcal{K}_n^{\pm}(\varphi))$ for all n.

Let us write $dd^c(\varphi) = T - \theta$ where T is a positive closed $(1,1)$ current, and θ is a smooth closed $(1,1)$ form. By property 4) of Definition 3.1, there is a strictly positive closed smooth $(1,1)$ form Θ so that $\Theta \geq \mathcal{K}_n^{\pm}(\theta)$ for any n. Then $f^*(\mathcal{K}_n^{\pm}(\varphi))$ are negative C^2 forms so that

$$\begin{aligned} dd^c f^*(\mathcal{K}_n^{\pm}(\varphi)) &= f^*(\mathcal{K}_n^{\pm}(dd^c\varphi)) = f^*(\mathcal{K}_n^{\pm}(T - \theta)) \\ &\geq f^*(\mathcal{K}_n^{\pm}(-\theta)) \geq -f^*(\Theta) \end{aligned}$$

for any n, i.e they are negative $f^*(\Theta)$-plurisubharmonic functions. Moreover the sequence of currents $f^*(\mathcal{K}_n^{\pm}(\varphi))$ has uniformly bounded mass (see the proof of Theorem 3.5). Therefore, by the compactness of this class of functions (see Chapter 1 in [**23**]), after passing to a subsequence if needed, we can assume that $f^*(\mathcal{K}_n^{\pm}(\varphi))$ converges in L^1 to negative functions denoted by $f^*(\varphi^{\pm})$. Let $S^{\pm}$ be any cluster points of $S_n^{\pm}$. Then $0 \geq S^{\pm} \geq f^*(\varphi^{\pm})$, which shows that any cluster point $S = S^+ - S^-$ of S_n has no mass on sets of Lebesgue measure zero. Hence to show that S is uniquely defined, it suffices to show that S is uniquely defined outside a proper analytic subset of Y.

Let E be a proper analytic subset of Y so that $f : X - f^{-1}(E) \to Y - E$ is a holomorphic submersion. If α is a smooth measure whose support is compactly contained in $X - f^{-1}(E)$ then $f_*(\alpha)$ is a smooth measure on Y. Hence by condition 4) of Definition 3.1, $\mathcal{K}_n(f_*(\alpha))$ uniformly converges to the smooth measure $f_*(\alpha)$.

Then it follows from the definition of S that

$$< S, \alpha >= \int_Y \psi \wedge f_*(\alpha).$$

Hence S is uniquely defined on $X - f^{-1}(E)$, and thus it is uniquely defined on the whole X, as wanted. □

The pullback of positive dd^c closed $(1,1)$ currents were defined by Alessandrini - Bassanelli [**2**] and Dinh -Sibony [**43**] under several contexts. Our definition coincides with theirs in the case of compact Kähler manifolds

Corollary 3.3. *Let X and Y be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Let T be a positive dd^c- closed $(1,1)$-current on Y. Then $f^*(T)$ is well-defined, and coincides with the usual definition.*

Proof. Consider a desingularization $\widetilde{\Gamma_f}$ and $\pi : \widetilde{\Gamma_f} \to X$ and $g : \widetilde{\Gamma_f} \to Y$ as in Theorem 3.2. Then it suffices to show that $g^*(T)$ is well-defined. This later follows from the proof of Theorem 5.5 in [**43**]. □

For a map $f : \mathbb{P}^k \to \mathbb{P}^k$, Russakovskii and Shiffman [**67**] defined the pullback of a linear subspace V of codimension p in $\mathbb{P}^k$ for which $\pi_2^{-1}(V) \cap \Gamma_f$ has codimension $\geq p$ in Γ_f. It can be easily seen that this is a special case of Corollary 3.1. In the same paper, we also find a definition for pullback of a measure having no mass on $\pi_Y(\mathcal{C}_f)$. Our definition coincides with theirs

Theorem 3.8. *Let X and Y be two compact Kähler manifolds. Let $f : X \to Y$ be a dominant meromorphic map. Let T be a positive measure having no mass on $\pi_Y(\mathcal{C}_f)$. Then $f^*(T)$ is well-defined, and coincides with the usual definition. Moreover, if T has no mass on proper analytic subvarieties of Y, then $f^*(T)$ has no mass on proper analytic subvarieties of X.*

PROOF. Let T be a positive measure on Y having no mass on $\pi_Y(\mathcal{C}_f)$. Let $\mathcal{K}_n$ be a good approximation scheme by C^2 forms. Then we will show that as n converges to ∞, any limit point of $[\Gamma_f]\wedge\pi_Y^*(\mathcal{K}_n(T))$ has no mass on $\mathcal{C}_f$. Thus $\lim_{n\to\infty}[\Gamma_f]\wedge\pi_Y^*(\mathcal{K}_n(T)) = (\pi_Y|_{\Gamma_f})^*(T)$ where the RHS is defined in [**43**]. Then $f^*(T)$ is well-defined, and moreover equals to the current $f^o(T)$ defined in [**43**], thus satisfies all the conclusions of Theorem 3.8.

Now we proceed to prove that any limit point τ of $[\Gamma_f]\wedge\pi_Y^*(\mathcal{K}_n(T))$ has no mass on $\mathcal{C}_f$. This is equivalent to showing that for a smooth $(dim(X),dim(X))$ form α on $X\times Y$, and for a sequence θ_j of smooth functions on $X\times Y$ having the properties: $0\leq\theta_j\leq 1$, $\theta_j=1$ on a neighborhood of $\mathcal{C}_f$, and support of θ_j converges to $\mathcal{C}_f$ then:

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{X\times Y}\theta_j\alpha\wedge[\Gamma_f]\wedge\pi_Y^*(\mathcal{K}_n(T))=0.$$

By properties of good approximation schemes by C^2 forms, we can write the above equality as

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{X\times Y}T\wedge\mathcal{K}_n((\pi_Y)_*(\theta_j\alpha\wedge[\Gamma_f]))=0. \tag{5}$$

Writing α as the difference of two positive smooth forms, we may assume that α is positive. Now α is a positive smooth form, since $0\leq\theta_j\leq 1$ for all j, we can bound the function $(\pi_Y)_*(\theta_j\alpha\wedge[\Gamma_f])$ by a multiplicity of $(\pi_Y)_*(\omega_{X\times Y}^{dim(X)}\wedge[\Gamma_f])$ independently of j. The later is a constant, thus $(\pi_Y)_*(\theta_j\alpha\wedge[\Gamma_f])$ is a positive bounded function. Then $\mathcal{K}_n((\pi_Y)_*(\theta_j\alpha\wedge[\Gamma_f]))$ are C^2 functions uniformly bounded w.r.t. j and n. Moreover, the support of $\mathcal{K}_n((\pi_Y)_*(\theta_j\alpha\wedge[\Gamma_f]))$ converges to $\pi_Y(\mathcal{C}_f)$ as $j\to\infty$, independent of n. Because T has no mass on $\pi_Y(\mathcal{C}_f)$, we can then apply Lebesgue's dominated convergence theorem to obtain (5). □

Alessandrini - Bassanelli [**2**] also defined the strict transform of a positive plurisubharmonic current of bidimension $(1,1)$ (i.e. a positive current T of bidimension (1,1) so that $dd^cT\geq 0$) by a modification. We can extend this to all bidimensions and all meromorphic maps. Given a DSH current T on Y, we say that the strict transform of T by a dominant meromorphic map $f : X \to Y$ is well-defined if the current $[\Gamma_f - \mathcal{C}_f] \wedge \pi_Y^*(T)$ (which is a well-defined current on $X \times Y - \mathcal{C}_f$, see Dinh and Sibony [**43**]) has locally finite mas across $\mathcal{C}_f$. In this case, we define the strict transform of the current T to be the push-forward by π_X of the trivial extension of $[\Gamma_f - \mathcal{C}_f] \wedge \pi_Y^*(T)$ to $X \times Y$.

THEOREM 3.9. *Let $f : X \to Y$ be a dominant meromorphic map between compact Kähler manifolds. Let T be a positive pseudo-pluri-subharmonic current, that is T is a positive current and $dd^cT \geq -\gamma$ for some smooth form γ. Then the strict transform of T is well-defined.*

PROOF. If $dd^c(T) \geq -\gamma$ where γ is a smooth form, then by the properties of good approximation schemes, there is a smooth form $\widetilde{\gamma}$ so that $dd^c\mathcal{K}_n^{\pm}(T) \geq -\widetilde{\gamma}$ for all n. Then $dd^c\pi_Y^*[\mathcal{K}_n^{\pm}(T)] \geq -\pi_Y^*(\widetilde{\gamma})$ for all n. Let (p,p) be the bidegree of T. By Lemma 3.4, there is a closed smooth form α and a strongly negative current S so that $[\Gamma_f] \wedge (\pi_X^*(\omega_X) + \pi_Y^*(\omega_Y))^{dim(Y)-p} \leq \alpha + dd^cS$. Therefore, the masses of the positive currents $[\Gamma_f] \wedge \pi_Y^*\mathcal{K}_n^{\pm}(T)$, which are

$$\begin{aligned}
& \int_{X\times Y} [\Gamma_f] \wedge (\pi_X^*(\omega_X) + \pi_Y^*(\omega_Y))^{dim(Y)-p} \wedge \pi_Y^*\mathcal{K}_n^{\pm}(T) \\
\leq & \int_{X\times Y} \alpha \wedge \pi_Y^*\mathcal{K}_n^{\pm}(T) + \int_{X\times Y} S \wedge dd^c\pi_Y^*\mathcal{K}_n^{\pm}(T) \\
\leq & \int_{X\times Y} \alpha \wedge \pi_Y^*\mathcal{K}_n^{\pm}(T) + \int_{X\times Y} S \wedge \pi_Y^*(-\widetilde{\gamma}),
\end{aligned}$$

are uniformly bounded. Therefore, the current $[\Gamma_f - \mathcal{C}_f] \wedge \pi_Y^*(T)$ (which is the limit on $\Gamma_f - \mathcal{C}_f$ of $[\Gamma_f] \wedge \pi_Y^*(\mathcal{K}_n^+(T) - \mathcal{K}_n^-(T))$, see [**43**]) has locally bounded mass across $\mathcal{C}_f$. By definition, the strict transform of T by the map f is well-defined. $\square$

4. Applications: invariant currents

4.1. Generalities on invariant currents. Throughout this subsection, we let X be a compact Kähler manifold of dimension k, and let $f : X \to X$ be a dominant meromorphic map. Let T be a current on X which can be pulled back by f. We say that T is an invariant current (in the broad sense) for f if there is a number λ so that $f^*(T) = \lambda T$. We have a similar definition when using the push forward instead of the pullback. Let $\delta_p(f)$ be the $p-th$ dynamical degree of f (see the preliminary chapter).

The map f is called p-algebraic stable (see, for example [**44**]) if $(f^*)^n = (f^n)^*$ as linear maps on $H^{p,p}(X)$ for all $n = 1, 2, \ldots$. When this condition is satisfied, it follows that $\delta_p(f) = r_p(f)$, thus helps in determining the p-th dynamical degree of f.

There is also the related condition of p-analytic stable (see [**44**]) which requires that

1) $(f^n)^*(T)$ is well-defined for any positive closed (p, p) current T and any $n \geq 1$.

2) Moreover, $(f^n)^*(T) = (f^*)^n(T)$ for any positive closed (p, p) current T and any $n \geq 2$.

For any selfmap f then f is k-algebraic stable where k =dimension of X. If f is a surjective holomorphic self-map then it is a finite-to-one map, and hence is p-analytic stable for any p. Since $H^{p,p}(X)$ is generated by classes of positive closed smooth (p, p) forms, p-analytic stability implies p-algebraic stability. For the converse of this, we have the following observation

PROPOSITION 3.3. *If $\pi_1(\mathcal{C}_f)$ has codimension $\geq p$, then f is p-analytic stable iff it is p-algebraic stable and satisfies condition 1) above so that $(f^*)^n(\alpha)$ is positive closed for any positive closed smooth (p, p) form and for any $n \geq 1$. Hence* 1*-algebraic stability is the same as* 1*-analytic stability.*

PROOF. First, let α be a positive closed smooth (p,p) form. Then $(f^n)^*(\alpha)$ is a current with L^1 coefficients. Then the assumption that $(f^*)^n(\alpha)$ is a positive closed current and the fact that $(f^*)^n(\alpha) = (f^n)^*(\alpha)$ outside a proper analytic set imply that $(f^*)^n(\alpha) \geq (f^n)^*(\alpha)$. But by the p-algebraic stability, these currents have the same cohomology class and hence must be the same. Hence the conclusion of Remark 3.3 holds for positive closed smooth (p,p) forms.

Now let T be a positive closed (p,p) current and let n be a positive integer. By Definition 3.2, there are positive closed smooth (p,p) forms $T_j^{\pm}$ so that $||T_j^{\pm}||$ is uniformly bounded, $T_j^+ - T_j^-$ weakly converges to T, and

$$(f^n)^*(T) = \lim_{j\to\infty}(f^n)^*(T_j^+ - T_j^-).$$

By the first paragraph of the proof $(f^n)^*(T_j^+ - T_j^-) = (f^*)^n(T_j^+ - T_j^-)$ for any n and j. Because $\pi_1(\mathcal{C}_f)$ has codimension $\geq p$, the continuity property in Theorem 3.5 implies that

$$\lim_{j\to\infty}(f^*)^n(T_j^+ - T_j^-) = (f^*)^n(T).$$

Therefore $(f^n)^*(T) = (f^*)^n(T)$ as wanted. □

A more general condition, called dd^c-p stability, seems to be natural for the problem of finding invariant (p,p) currents for a self-map f.

DEFINITION 3.3. *We say that f satisfies the dd^c-p stability condition if the following holds: For any smooth $(p-1,p-1)$ form α and for any n, $f^*((f^n)^*dd^c\alpha)$ is well-defined, and moreover $f^*((f^n)^*dd^c\alpha) = (f^{n+1})^*(dd^c\alpha)$.*

In general, condition of dd^c-p stability has no relation with condition of p-algebraic stability. On the one hand, the dd^c-p stability condition requires no constraints on the action of f^* on $H^{p,p}(X)$, because the cohomology class of $dd^c(\alpha)$ is zero. On the other hand, it asks for the possibility of iterated pull-back $dd^c(\alpha)$ by f. Any map f

is dd^c-1 stable, whether being or not 1-algebraic stable. If f is p-analytic stable then f is dd^c-p stable. Using the method in Step 1 of the proof of Lemma 3.8, it can be shown that the linear pseudo-automorphisms in [**5**] are dd^c-2 stable. We suspect that these pseudo-automorphisms are also 2-analytic stable.

LEMMA 3.5. *Assume that f satisfies the dd^c-p stability condition. Let λ be a positive real number. If $|\lambda| > \delta_{p-1}(f)$, then for any smooth $(p-1,p-1)$ form α, there is a current R_α of order 0, so that $f^*(dd^c R_\alpha)$ is well-defined, and moreover*

$$f^*(dd^c R_\alpha) - \lambda dd^c R_\alpha = \lambda dd^c \alpha.$$

PROOF. Define $\beta = -\alpha$, and consider

$$R_n = \sum_{j=0}^{n} \frac{(f^j)^*(\beta)}{\lambda^j}.$$

Since β is a smooth $(p-1,p-1)$ form, there is a constant $A > 0$ so that $-A\omega_X^{p-1} \leq \beta \leq A\omega_X^{p-1}$. It follows that

$$R_\alpha = \sum_{j=0}^{\infty} \frac{(f^j)^*(\beta)}{\lambda^j}$$

is a well-defined current which is a difference of two positive currents, hence of order 0. Moreover $-S_n \leq R_n - R \leq S_n$, where

$$S_n = A \sum_{j=n+1}^{\infty} \frac{(f^j)^*(\omega_X^{p-1})}{|\lambda|^j}.$$

The S_n are well-defined positive closed $(p-1,p-1)$ currents, because it is well-known (see for example Chapter 2 in [**56**]) that

$$\lim_{n\to\infty} ||(f^n)^*(\omega_X^{p-1})||^{1/n} = \delta_{p-1}(f),$$

and the latter is $< |\lambda|$ by assumption. The above inequality also shows that $||S_n|| \to 0$ as $n \to \infty$. The dd^c-p stability condition shows that $f^*(dd^c R_n)$ is well-defined for any

n, and moreover $f^*(dd^cR_n) - \lambda dd^cR_{n+1} = -\lambda dd^c\beta = \lambda dd^c\alpha$. Applying Theorem 3.4, using that R_n weakly converges to R_α, we have

$$f^*(dd^cR_\alpha) - \lambda dd^cR_\alpha = \lambda dd^c\alpha.$$

□

We have the following abstract result on invariant (p,p) currents.

THEOREM 3.10. *Assume that $f : X \to X$ satisfies the dd^c-p stability condition. Let λ be a real eigenvalue of $f^* : H^{p,p}(X) \to H^{p,p}(X)$, and let $0 \neq \theta_\lambda \in H^{p,p}(X)$ be an eigenvector with eigenvalue λ. Assume moreover that $|\lambda| > \delta_{p-1}(f)$ and let $s \geq 2$ be an integer. Then any of the following statements is equivalent to each other:*

1) There is a closed (p,p) current T of order s with $\{T\} = \theta_\lambda$ so that $f^*(T)$ is well-defined, and moreover $f^*(T) = \lambda T$.

2) There are a smooth $(p-1,p-1)$ form α and a closed (p,p) current T of order s with $\{T\} = \theta_\lambda$ so that $f^*(T)$ is well-defined, and moreover $f^*(T) = \lambda T + \lambda dd^c(\alpha)$.

3) For any smooth $(p-1,p-1)$ form α, there is a closed (p,p) current T of order s with $\{T\} = \theta_\lambda$ so that $f^*(T)$ is well-defined, and moreover $f^*(T) = \lambda T + \lambda dd^c(\alpha)$.

4) There is a closed (p,p) current T of order s with $\{T\} = \theta_\lambda$ so that $f^*(T)$ is well-defined, and moreover $f^*(T) - \lambda T$ is a smooth form.

PROOF. All of the equivalences follow easily from Lemma 3.5.

1) $\Rightarrow$ 3): Let T_0 be a closed (p,p) current of order s with $\{T_0\} = \theta_\lambda$ so that $f^*(T_0)$ is well-defined, and $f^*(T_0) - \lambda T_0 = 0$. For any smooth $(p-1,p-1)$ form α on X, let R_α be the current constructed in Lemma 3.5. Then $T = T_0 + dd^c(R_\alpha)$ is a closed (p,p) current of order s with $\{T\} = \theta_\lambda$ so that $f^*(T)$ is well-defined, and $f^*(T) - \lambda T = dd^c(R_\alpha)$.

3) $\Rightarrow$ 2: Obviously.

2) $\Rightarrow$ 1): Let α_0 be a smooth $(p-1,p-1)$ form, and let T_0 be a closed (p,p) current of order s with $\{T_0\}=\theta_\lambda$ so that $f^*(T_0)$ is well-defined, and $f^*(T_0)-\lambda T_0=dd^c(\alpha_0)$. Let R_α be the current constructed in Lemma 3.5. Then $T=T_0-dd^c(R_\alpha)$ is a closed (p,p) current of order s with $\{T\}=\theta_\lambda$ so that $f^*(T)$ is well-defined, and $f^*(T)-\lambda T=0$.

Finally, that 2) and 4) are equivalent follows from the dd^c lemma, since the current $f^*(T)-\lambda T$ is a smooth form cohomologous to 0. □

4.2. Applications. Now we give some explicit examples on the existence of invariant currents.

LEMMA 3.6. *Let X be a compact Kähler manifold with a Kähler form ω_X and $f:X\to X$ be a dominant meromorphic map. Assume that $\pi_1(\mathcal{C}_f)$ has codimension $\geq p$ and f is p-analytic stable. Let $0\neq\theta$ be an eigenvector with respect to the eigenvalue $\lambda=r_p(f)$ the spectral radius of the linear map $f^*:H^{p,p}(X)\to H^{p,p}(X)$. Assume moreover that $||(f^n)^*(\omega_X^p)||\sim\lambda^n$ as $n\to\infty$. Then there is a closed (p,p) current T which is a difference of two positive closed (p,p) currents, and has the properties that $\{T\}=\theta$ and $f^*(T)=\lambda T$.*

Remark: Since f is p-analytic stable, the condition on $||(f^n)^*(\omega_X^p)||$ can be easily checked by looking at the Jordan form for f^* (see e.g. [**56**]). Variants of Lemma 3.6 are also available. Lemma 3.6 generalizes the results for the standard case $p=1$, and for the case $X=\mathbb{P}^k$ in Dinh and Sibony [**44**]. Lemma 3.6 can be applied to the pseudo-automorphism J_X in Subsection 5.1 for $p=2$, and may be applicable to many other pseudo-automorphisms e.g. those in [**5**]. The method of the proof of Lemma 3.8 may be used to check the p-analytic stability of a meromorphic self-map.

PROOF. Since $\pi_1(\mathcal{C}_f)$ has codimension $\geq p$, it follows from Theorem 3.5 any positive closed (p,p) current can be pulled back, and the pullback operator is continuous with respect to the weak topology on positive closed (p,p) currents. We can represent θ by a difference $\alpha = \alpha^+ - \alpha^-$ of two positive closed smooth (p,p) forms $\alpha^\pm$. Since f is p-analytic stable, it follows that $(f^n)^*(\alpha^\pm) = (f^*)^n(\alpha^\pm)$ are positive closed (p,p) currents for any $n \geq 1$. Moreover there is a constant $C_1 > 0$ so that $||(f^*)^n(\alpha^\pm)|| = ||(f^n)^*(\alpha^\pm)|| \leq C_1 r_p(f)^n = C_1\lambda^n$ (see e.g [**56**]). We follow the standard construction of an invariant current under these assumptions (see [**68**] and [**19**]). Consider the currents $T_N = T_N^+ - T_N^-$, where

$$T_N^\pm = \frac{1}{N}\sum_{j=0}^{N-1}\frac{(f^*)^j(\alpha^\pm)}{\lambda^j}.$$

Then $T_N^\pm$ are positive closed (p,p) currents with uniformly bounded masses, thus after passing to a subsequence, we may assume that they converge to $T^\pm$. We define $T = T^+ - T^-$. Since $\{T_N\} = \{\alpha\}$ for any N, we also have $\{T\} = \{\alpha\}$. Since $f^*(T_N^\pm) - \lambda T_N^\pm$ converges to 0, it follows that $f^*(T) = \lambda T$. $\square$

Let us continue with an application concerning invariant positive closed currents whose supports are contained in pluripolar sets.

Corollary 3.4. *Let $f_1 : \mathbb{P}^{k_1} \rightarrow \mathbb{P}^{k_1}$ and $f_2 : \mathbb{P}^{k_2} \rightarrow \mathbb{P}^{k_2}$ be dominant rational maps not 1-algebraic stable, of degrees d_1 and d_2 respectively. Then there is a nonzero positive closed $(2,2)$ current T on $\mathbb{P}^{k_1} \times \mathbb{P}^{k_2}$ with the following properties:*

1) $f^*(T)$ is well-defined and moreover $f^*(T) = d_1 d_2 T$, here $f = f_1 \times f_2$.

2) The support of T is pluripolar.

The existence of Green currents T_1 and T_2 for f_1 and f_2 were proved by Sibony [**68**] (see also [**19**]). The current T is in fact the product $T_1 \times T_2$. Its support is contained in a countable union of analytic varieties of codimension 2 in $\mathbb{P}^{k_1} \times \mathbb{P}^{k_2}$.

Proof. Let T_1 and T_2 be the Green $(1,1)$ currents for the maps f_1 and f_2 as constructed in Sibony [**68**], respectively. Then we can write

$$T_i = \sum_j \lambda_{j,i}[V_{j,i}]$$

for $i = 1, 2$, where $\lambda_{j,i} > 0$ and $V_{j,i}$ are irreducible hypersurfaces in $\mathbb{P}^{k_i}$. Moreover $f^*(T_1) = d_1 T_1$ and $f^*(T_2) = d_2 T_2$. We choose $T = T_1 \times T_2$. Consider the finite summands

$$S_{N,i} = \sum_{j=0}^{N} \lambda_{j,i}[V_{j,i}].$$

Then $f^{-1}(S_{N,1} \times S_{N,2}) = f_1^{-1}(S_{N,1}) \times f_2^{-1}(S_{N,2})$ has codimension 2 in $\mathbb{P}^{k_1} \times \mathbb{P}^{k_2}$, thus $f^*(S_{N,1} \times S_{N,2})$ are well-defined by Corollary 3.1. Since $T_1 \times T_2 - S_{N,1} \times S_{N,2}$ are positive closed currents decreasing to 0, it follows by Theorem 3.4 that $f^*(T_1 \times T_2)$ is well-defined and moreover

$$f^*(T_1 \times T_2) = \lim_{N \to \infty} f^*(S_{N,1} \times S_{N,2}).$$

It remains to show that $f^*(T_1 \times T_2) = d_1 d_2 T_1 \times T_2$. To this end, first we show that $f^*(S_{N,1} \times S_{N,2}) = f_1^*(S_{N,1}) \times f_2^*(S_{N,2})$ for any N. By the results in [**44**] (see also the last section), there are positive closed $(1,1)$ currents $W_{j,N,1}$ on $\mathbb{P}^{k_1}$ and $W_{j,N,2}$ on $\mathbb{P}^{k_2}$ with uniformly bounded norms so that $S_{N,1} = \lim_{j\to\infty} W_{j,N,1}$ and $S_{N,2} = \lim_{j\to\infty} W_{j,N,2}$. Moreover, we can choose these approximations in such a way that support of $W_{j,N,1}$ converges to $S_{N,1}$ and support of $W_{j,N,2}$ converges to $S_{N,2}$. Then $\lim_{j\to\infty} W_{j,N,1} \times W_{j,N,2} = S_{N,1} \times S_{N,2}$, and $W_{j,N,1} \times W_{j,N,2}$ has uniformly bounded mass and locally uniformly converges to 0 on $\mathbb{P}^{k_1} \times \mathbb{P}^{k_2} - S_{N,1} \times S_{N,2}$. Hence we can apply Theorem 3.6 to obtain that

$$\begin{aligned} f^*(S_{N,1} \times S_{N,2}) &= \lim_{j\to\infty} f^*(W_{j,N,1} \times W_{j,N,2}) = \lim_{j\to\infty} f_1^*(W_{j,N,1}) \times f_2^*(W_{j,N,2}) \\ &= f_1^*(S_{N,1}) \times f_2^*(S_{N,2}). \end{aligned}$$

Having this, it follows from the continuity of pullback on positive closed $(1,1)$ currents and the definitions of T_1 and T_2 that

$$\begin{aligned} f^*(T_1 \times T_2) &= \lim_{N\to\infty} f^*(S_{N,1} \times S_{N,2}) = \lim_{N\to\infty} f_1^*(S_{N,1}) \times f_2^*(S_{N,2}) \\ &= f_1^*(T_1) \times f_2^*(T_2) = d_1 d_2 T_1 \times T_2. \end{aligned}$$

□

Corollary 3.5. *Let X be a compact Kähler manifold of dimension k, and let $f : X \to X$ be a dominant meromorphic map. Assume that f has large topological degree, i.e. $\delta_k(f) > \delta_{k-1}(f)$. Then f has an invariant positive measure μ, i.e. $f^*(\mu) = \delta_k(f)\mu$.*

The result of Corollary 3.5 belongs to Guedj [**57**] and Dinh-Sibony [**41**], who showed in addition that the measure μ has no mass on proper analytic subvarieties.

Proof. It is well-known that for any smooth (k,k) form θ then $(f^n)^*(\theta) = (f^*)^n(\theta)$ for all n (see for example [**57**] or Theorem 3.8). Hence f satisfies dd^c-k stability condition. As in [**57**], we can find a smooth probability measure θ so that $f^*(\theta)$ is again a smooth probability measure. Hence $f^*(\theta) - \delta_k(f)\theta = dd^c(\varphi)$, where φ is a smooth $(p-1,p-1)$ form. Hence we can apply Theorem 3.10. □

Corollary 3.6. *Let X be a compact Kähler manifold, and let $f : X \to X$ be a surjective holomorphic map. Let λ be a real eigenvalue of $f^* : H^{p,p}(X) \to H^{p,p}(X)$, and let $0 \neq \theta_\lambda \in H^{p,p}(X)$ be an eigenvector with eigenvalue λ. Assume moreover that $|\lambda| > \delta_{p-1}(f)$. Then there is a closed current T of order 2 with $\{T\} = \theta_\lambda$ so that $f^*(T)$ is well-defined, and moreover $f^*(T) = \lambda T$.*

Proof. Let θ be a smooth form then $f^*(\theta)$ is again a smooth form since f is holomorphic. Then we can use the same arguments as that in the proof of Corollary 3.5. □

Note that in Corollary 3.6 we do not require that $\lambda = \delta_p$ or that the cohomology class θ_λ is presented by a positive closed current.

Example 3: Let $X = \mathbb{P}^2_{w_1} \times \mathbb{P}^2_{w_2} \times \mathbb{P}^2_{w_3}$, and let $f : X \to X$ to be $f(w_1, w_2, w_3) = (P_2(w_2), P_3(w_3), P_1(w_1))$ where $P_1, P_2, P_3 : \mathbb{P}^2 \to \mathbb{P}^2$ are surjective holomorphic maps of degrees ≥ 2, and not all of them are submersions (For example, we can choose one of them to be $P[z_0 : z_1 : z_2] = [z_0^d : z_1^d : z_2^d]$ for some integer $d \geq 2$). Corollary 3.6 can be applied to find invariant currents for f.

5. The map J_X and open questions

5.1. The map J_X. Through out this subsection, let X be the blowup of $\mathbb{P}^3$ along 4 points $e_0 = [1 : 0 : 0 : 0], e_1 = [0 : 1 : 0 : 0], e_2 = [0 : 0 : 1 : 0], e_3 = [0 : 0 : 0 : 1]$; $J : \mathbb{P}^3 \to \mathbb{P}^3$ is the Cremona map $J[x_0 : x_1 : x_2 : x_3] = [1/x_0 : 1/x_1 : 1/x_2 : 1/x_3]$, and let J_X be the lifting of J to X.

Remark 3.2. *The map J_X was given in Example 2.5 page 33 in* [**56**] *where the author showed that the map $J_X^* : H^{2,2}(X) \to H^{2,2}(X)$ does not preserve the cone of cohomology classes generated by positive closed $(2,2)$ currents.*

For $0 \leq i \neq j \leq 3$, $\Sigma_{i,j}$ is the line in $\mathbb{P}^3$ consisting of points $[x_0 : x_1 : x_2 : x_3]$ where $x_i = x_j = 0$, and $\widetilde{\Sigma_{i,j}}$ is the strict transform of $\Sigma_{i,j}$ in X.

Let E_0, E_1, E_2, E_3 be the corresponding exceptional divisors of the blowup $X \to \mathbb{P}^3$, and let L_0, L_1, L_2, L_3 be any lines in E_0, E_1, E_2, E_3 correspondingly. Let H be a generic hyperplane in $\mathbb{P}^3$, and let H^2 be a generic line in $\mathbb{P}^3$. Then H, E_0, E_1, E_2, E_3 are a basis for $H^{1,1}(X)$, and H^2, L_0, L_1, L_2, L_3 are a basis for $H^{2,2}(X)$. Intersection products in complementary dimensions are (see for example Chapter 4 in [**54**]):

$$H.H^2 = 1,\ H.L_0 = 0,\ H.L_1 = 0,\ H.L_2 = 0,\ H.L_3 = 0,$$

$$E_0.H^2 = 0,\ E_0.L_0 = -1,\ E_0.L_1 = 0,\ E_0.L_2 = 0,\ E_0.L_3 = 0,$$

$$E_1.H^2 = 0,\ E_1.L_0 = 0,\ E_1.L_1 = -1,\ E_1.L_2 = 0,\ E_1.L_3 = 0,$$

$$E_2.H^2 = 0,\ E_2.L_0 = 0,\ E_2.L_1 = 0,\ E_2.L_2 = -1,\ E_1.L_3 = 0,$$

$$E_3.H^2 = 0,\ E_3.L_0 = 0,\ E_3.L_1 = 0,\ E_3.L_2 = 0,\ E_3.L_3 = -1.$$

The map $J_X^* : H^{1,1}(X) \to H^{1,1}(X)$ is not hard to compute (see for example the computations in Example 2.5 in [**56**]). We let Σ_i be the linear subspace of $\mathbb{P}^3$ defined by the equation $z_i = 0$, and let $\widetilde{\Sigma}_i$ be the strict transform of Σ in X. Then since $J_X^{-1}(E_i) = \widetilde{\Sigma}_i$, $J_X^{-1}(\widetilde{\Sigma}_i) = E_i$ and the class of H is $\pi^*(\Sigma_i) = \widetilde{\Sigma}_i + \sum_{j\neq i} E_j$, we find

$$\begin{aligned}
J_X^*(H) &= 3H - 2E_0 - 2E_1 - 2E_2 - 2E_3,\\
J_X^*(E_0) &= H - E_1 - E_2 - E_3,\\
J_X^*(E_1) &= H - E_0 - E_2 - E_3,\\
J_X^*(E_2) &= H - E_0 - E_1 - E_3,\\
J_X^*(E_3) &= H - E_0 - E_1 - E_2.
\end{aligned}$$

If $x \in H^{1,1}(X)$ and $y \in H^{2,2}(X)$, since J_X^2 =the identity map on X, we have the duality $(J_X^* y).x = y.(J_X^* x)$. Thus from the above data, we can write down the map $J_X^* : H^{2,2}(X) \to H^{2,2}(X)$:

$$\begin{aligned}
J_X^*(H^2) &= 3H^2 - L_0 - L_1 - L_2 - L_3,\\
J_X^*(L_0) &= 2H^2 - L_1 - L_2 - L_3,\\
J_X^*(L_1) &= 2H^2 - L_0 - L_2 - L_3,\\
J_X^*(L_2) &= 2H^2 - L_0 - L_1 - L_3,\\
J_X^*(L_3) &= 2H^2 - L_0 - L_1 - L_2.
\end{aligned}$$

COROLLARY 3.7. *For* $0 \leq i \neq j \leq 3$, *let* $\Sigma_{i,j}$ *be the line in* $\mathbb{P}^3$ *consisting of points* $[x_0 : x_1 : x_2 : x_3]$ *where* $x_i = x_j = 0$. *Let* $\widetilde{\Sigma_{i,j}}$ *be the strict transform of* $\Sigma_{i,j}$ *in* X.

For any positive closed $(2,2)$ *current* T, $J_X^*(T)$ *is well-defined. Moreover,* $J_X^*([\widetilde{\Sigma_{0,1}}]) = -[\widetilde{\Sigma_{2,3}}]$ *and* $J_X^*([\widetilde{\Sigma_{2,3}}]) = -[\widetilde{\Sigma_{0,1}}]$.

PROOF. The restriction $J_X : X - \bigcup \widetilde{\Sigma_{i,j}} \to X - \bigcup \widetilde{\Sigma_{i,j}}$ is a biholomorphic map, as can be seen by using local coordinate projections for the blowup π near the exceptional divisors E_i's. Moreover it can be shown that $J_X(\widetilde{\Sigma_{i,j}}) = \widetilde{\Sigma_{3-i,3-j}}$, and every point on $\widetilde{\Sigma_{i,j}}$ blows up to $\widetilde{\Sigma_{3-i,3-j}}$. Hence $\pi_1(\mathcal{C}_{J_X}) = \bigcup \widetilde{\Sigma_{i,j}}$. Therefore the map J_X satisfies Theorem 3.5 for $p = 2$. Thus if T is a positive closed $(2,2)$ current on X then $J_X^*(T)$ is well-defined. For an alternative proof of this fact, see Lemma 3.7 below.

It remains to show that $J_X^*[\widetilde{\Sigma_{0,1}}] = -[\widetilde{\Sigma_{2,3}}]$. Since $J_X^{-1}(\widetilde{\Sigma_{0,1}}) = \widetilde{\Sigma_{2,3}}$, by Theorem 3.6 there is a number λ so that $J_X^*[\widetilde{\Sigma_{0,1}}] = \lambda[\widetilde{\Sigma_{2,3}}]$. To determine λ, we need to know $J_X^*\{\widetilde{\Sigma_{0,1}}\}$. We have $\{\widetilde{\Sigma_{0,1}}\} = \{H^2 - L_2 - L_3\}$, hence from the above data we have

$$J_X^*\{\widetilde{\Sigma_{0,1}}\} = J_X^*\{H^2\} - J_X^*\{L_2\} - J_X^*\{L_3\} = \{-H^2 + L_0 + L_1\} = -\{\widetilde{\Sigma_{2,3}}\},$$

thus $\lambda = -1$, and $J_X^*[\widetilde{\Sigma_{0,1}}] = -[\widetilde{\Sigma_{2,3}}]$. $\square$

The following result gives an alternative proof to the conclusions of Corollary 3.7. In its proof we will make use of the space Y defined in the statement of Proposition 3.4 below. Here $\pi : Y \to X$ is the blowup of X along all submanifolds $\widetilde{\Sigma_{i,j}}$ $(1 \leq i < j \leq 3)$. Then the lifting map J_Y of J to Y is an involutive automorphism. Moreover, if we let $S_{i,j}$ denote the exceptional divisor of Y over $\widetilde{\Sigma_{i,j}}$, then $J_Y(S_{0,1}) = S_{2,3}$, $J_Y(S_{0,2}) = S_{1,3}$, and $J_Y(S_{0,3}) = S_{1,2}$.

LEMMA 3.7. *Let* T_n^+ *and* T_n^- *be positive closed smooth* $(2,2)$ *forms on* X, *so that*

i) $||T_n^+||$, $||T_n^-||$ *are uniformly bounded,*

and

ii) $T_n^+ - T_n^- \rightharpoonup [\widetilde{\Sigma_{0,1}}]$.

Then $J_X^*(T_n^+ - T_n^-) \rightharpoonup -[\widetilde{\Sigma_{2,3}}]$.

As a consequence, if we replace $[\widetilde{\Sigma_{0,1}}]$ *in i) and ii) above by any positive closed* $(2,2)$ *current* T *then* $J_X^*(T_n^+ - T_n^-)$ *converges to* $J_X^*(T)$.

Proof. Let $\tau_n^+ = \pi^*(T_n^+)$ and $\tau_n^- = \pi^*(T_n^-)$, which are positive closed currents on Y. By assumption i), $||\tau_n^+||$ and $||\tau_n^-||$ are uniformly bounded. Thus we may assume that $\tau_n^+ \rightharpoonup \tau^+$ and $\tau_n^- \rightharpoonup \tau^-$, where τ^+ and τ^- are positive closed currents on Y.

Since J_Y is a biholomorphic map, we can pull-back any current on Y by J_Y. It is not hard to see that

$$\begin{aligned} J_X^*(T_n^+) &= \pi_*(J_Y^*\tau_n^+), \\ J_X^*(T_n^-) &= \pi_*(J_Y^*\tau_n^-). \end{aligned}$$

Hence

$$J_X^*(T_n^+ - T_n^-) \rightharpoonup \pi_*(J_Y^*(\tau^+ - \tau^-)).$$

We need to show that the latter current is $-[\widetilde{\Sigma_{2,3}}]$. To this end, it suffices to show that support of $\pi_*(J_Y^*(\tau^+ - \tau^-))$ is in $\widetilde{\Sigma_{2,3}}$. In fact, then we will have $\pi_*(J_Y^*(\tau^+ - \tau^-)) = \lambda[\widetilde{\Sigma_{2,3}}]$, and the computation on cohomology shows that $\lambda = -1$.

It is not hard to see that support of $\tau^+ - \tau^-$ is contained in the union of $S_{i,j}$'s ($1 \leq i < j \leq 3$). Let $\tau_{i,j} = \tau^+|_{S_{i,j}} - \tau^-|_{S_{i,j}}$ with support in $S_{i,j}$ so that $\tau = \sum_{1\leq i<j\leq 3} \tau_{i,j}$. In $H^{2,2}(Y)$ we have:

$$\pi^*\{\widetilde{\Sigma_{0,1}}\} = \{\tau^+ - \tau^-\} = \sum_{i,j} \{\tau_{i,j}\},$$

here $\pi^*\{\widetilde{\Sigma_{0,1}}\}$ can be represented by currents with support in $S_{0,1}$. Moreover, by considering the push-forwards $\pi_*(\tau_n^+ - \tau_n^-)$, it follows that $\pi_*(\tau_{i,j}) = 0$ where $(i,j) \neq (0,1)$. It can be checked that each fiber $S_{i,j}$ is a product $S_{i,j} \simeq \mathbb{P}^1 \times \mathbb{P}^1$, hence by Küneth's theorem $H^{2,2}(S_{i,j})$ is generated by a "horizontal curve" $\alpha_{i,j}$ and a "vertical curve" (or fiber) $\beta_{i,j}$. Here the properties of "horizontal curve" and "vertical curve" that we use are that $\pi_*(\alpha_{i,j})=\widetilde{\Sigma}_{i,j}$ and $\pi_*(\beta_{i,j})=0$. Hence there are numbers $a_{i,j}$ and $b_{i,j}$ so that the cohomology class of $\tau_{i,j}-a_{i,j}\alpha_{i,j}-b_{i,j}\beta_{i,j}$ is zero. For $(i,j)\neq(0,1)$, since $\pi_*(\tau_{i,j})=0$, it follows that

$$a_{i,j}\{\widetilde{\Sigma}_{i,j}\}=\pi_*\{a_{i,j}\alpha_{i,j}+b_{i,j}\beta_{i,j}\}=\pi_*\{\tau_{i,j}\}=\{\pi_*(\tau_{i,j})\}=0.$$

Hence $a_{i,j}=0$ for $(i,j)\neq(0,1)$.

Note that a non-zero $(2,2)$-cohomology class in $H^{2,2}(Y)$ represented by currents with supports in $S_{0,1}$ can not be represented by a linear combinations of "vertical curves" with support in $\bigcup_{(i,j)\neq(0,1)}S_{i,j}$: Assume that

$$\{a_{0,1}\alpha_{0,1}+b_{0,1}\beta_{0,1}+\sum_{(i,j)\neq(0,1)}b_{i,j}\beta_{i,j}\}=0$$

in $H^{2,2}(Y)$. Push-forward by the map π implies that $a_{0,1}\{\widetilde{\Sigma}_{0,1}\}=0$ in $H^{2,2}(X)$, and hence $a_{0,1}=0$. Thus $\{\sum b_{i,j}\beta_{i,j}\}=0$ in $H^{2,2}(Y)$. Use the fact that $\{S_{i,j}\}.\{\beta_{k,l}\}=-1$ if $(k,l)=(i,j)$, and $=0$ otherwise (see for example Chapter 4 in [**54**]), we imply that $b_{i,j}=0$ for all (i,j) as claimed.

Hence it follows that $\{\tau_{i,j}\}=0$ in $H^{2,2}(Y)$ for $(i,j)\neq(0,1)$.

We have

$$\pi_*(J_Y^*(\tau^+-\tau^-))=\sum_{i,j}\pi_*(J_Y^*\tau_{i,j}),$$

where support of $\pi_*(J_Y^*\tau_{i,j})$ is contained in $\widetilde{\Sigma_{3-i,3-j}}$. Here we use the convention that $\widetilde{\Sigma_{k,l}}:=\widetilde{\Sigma_{l,k}}$ if $k>l$. Since $\pi_*(J_Y^*\tau_{i,j})$ is a normal $(2,2)$ current, it follows from the structure theorem for normal currents that there is $\lambda_{i,j}\in\mathbb{R}$ so that $\pi_*(J_Y^*\tau_{i,j})=\lambda_{i,j}[\widetilde{\Sigma_{3-i,3-j}}]$. If $(i,j)\neq(0,1)$ then $\{\tau_{i,j}\}=0$ in $H^{2,2}(Y)$, thus $\{\pi_*(J_Y^*\tau_{i,j})\}=0$ in $H^{2,2}(X)$, which implies $\lambda_{i,j}=0$ for such (i,j)'s. Hence

$$\pi_*(J_Y^*(\tau^+-\tau^-))=\pi_*(J_Y^*\tau_{0,1})$$

has support in $\widetilde{\Sigma_{2,3}}$ as wanted. □

PROPOSITION 3.4. *Let X be the space constructed in Corollary 3.7. Let $\pi : Y \to X$ be the blowup of X along all submanifolds $\widetilde{\Sigma_{i,j}}$ ($1 \leq i < j \leq 3$). Then there is a positive closed $(2,2)$-current T on X with L^1 coefficients so that: in $H^{2,2}(Y)$,*

$$\{\pi^o(T)\} \neq \pi^*\{T\}.$$

Here the operator π^o is defined in Dinh and Nguyen [**35**]. *In this case, in fact $\pi^o(T)$ is also the operator defined in Dinh and Sibony* [**43**].

PROOF. We assume in order to reach a contradiction that for any positive closed $(2,2)$ current T on X with L^1-coefficients then $\{\pi^o(T)\} = \pi^*\{T\}$ in $H^{2,2}(Y)$.

By regularization theorem of Dinh and Sibony, there is a sequence T_n^+ and T^- of positive closed $(2,2)$ currents with L^1-coefficients such that $||T_n^+||$ are uniformly bounded and $T_n^+ \rightharpoonup T^- + [\widetilde{\Sigma_{0,1}}]$. By the assumption we have $\{\pi^o(T_n^+)\} = \pi^*\{T_n^+\}$ for any n, and $\{\pi^o(T^-)\} = \pi^*\{T^-\}$. Now for the maps J_X and J_Y considered above, it is not hard to see that $J_X^o = \pi_* J_Y^* \pi^o$. Thus, we also have $\{J_X^o(T_n^+)\} = J_X^*\{T_n^+\}$ and $\{J_X^o(T^-)\} = J_X^*\{T^-\}$.

Let τ^+ be a cluster point of $J_X^o(T_n^+)$. Then it is easy to see that

$$\tau^+ \geq J_X^o(T^- + [\widetilde{\Sigma_{0,1}}]) = J_X^o(T^-) + J_X^o([\widetilde{\Sigma_{0,1}}]) = J_X^o(T^-).$$

But then this contradicts the fact that in $H^{2,2}(X)$:

$$\begin{aligned} \{\tau^+\} &= \lim_{n\to\infty} \{J_X^o(T_n^+)\} = \lim_{n\to\infty} J_X^*\{(T_n^+)\} \\ &= J_X^*\{T^-\} + J_X^*\{\widetilde{\Sigma_{0,1}}\} = \{J_X^o(T^-)\} - \{[\widetilde{\Sigma_{2,3}}]\}, \end{aligned}$$

here we used the assumption that $J_X^*\{(T_n^+)\} = \{J_X^o(T_n^+)\}$ and $J_X^*\{(T^-)\} = \{J_X^o(T^-)\}$.

□

PROPOSITION 3.5. *Let X be the space constructed in Corollary 3.7. There is no sequence T_n^+ and T^- of positive closed smooth $(2,2)$ forms on X such that* *i)* $||T_n^+||$ *are uniformly bounded*

ii) $T_n^+ - T^- \rightharpoonup [\widetilde{\Sigma_{0,1}}]$.

REMARK 3.3. *In Example 6.3 of the paper* [**13**] *of Bost, Gillet, and Soule, a related result was given.*

PROOF. Use the same argument as that in the proof of Proposition 3.4, but now use that if $T_n^{\pm}$ are positive closed smooth forms then $J_X^*(T_n^{\pm}) = J_X^o(T_n^{\pm})$, and hence $\{J_X^o(T_n^{\pm})\} = J_X^*\{T_n^{\pm}\}$. □

We end this Subsection by showing that the map J_X is 2-analytic stable.

LEMMA 3.8. *J_X is 2-algebraic stable and $(J_X^*)^2 = Id = (J_X^2)^*$ on positive closed $(2,2)$ currents.*

PROOF. Since J_X has no exceptional hypersurface, J_X is 1-algebraic stable. Because $J_X = J_X^{-1}$, it follows by duality that J_X is also 2-algebraic stable. Since $J_X^2 = Id$, it remains to check that $(J_X^*)^2 = Id$. Define $A = \bigcup_{i\neq j} \widetilde{\Sigma}_{i,j}$.

1) First we show that for a DSH^1 current R then:

$$(J_X^*)^2(R) = R. \tag{6}$$

For this end, first we show that $(J_X^*)^2(R) = R$ on $X - A$. Since J_X^* is continuous in the DSH^1 topology by Theorem 3.5, using the approximation theorem for DSH currents it suffices to show (6) for a smooth $(1,1)$ form R. In that case it is easy to see, since $(J_X^*)^2(R)$ is determined by its restriction on $X - A$, and on $X - A$ it is not other than the usual pullback of smooth forms $(J_X|_{X-A}^*)^2(R)$.

Having $(J_X^*)^2(R) = R$ on $X - A$, then (6) follows by the Federer type of support in [**3**].

2) It follows from 1) that if T is a positive closed $(2,2)$ current on X, then $(J_X^*)^2(T) - T$ depends only on the cohomology class of T. In fact, if T' is a positive

closed $(2,2)$ current having the same cohomology class as T, then $T - T' = dd^c(R)$ for a DSH^1 current R. Then from 1)

$$(J_X^*)^2(T) - (J_X^*)^2(T') = dd^c(J_X^*)^2(R) = dd^c(R) = T - T'.$$

3) From 2), to prove Lemma 3.8 it suffices to show it for a set of positive closed currents whose cohomology classes generate $H^{2,2}(X)$. For such a set, we can consider the currents of integrations on a generic line in $\mathbb{P}^3$, a generic line in the exceptional divisors E_0, E_1, E_2, E_3, and the line $\widetilde{\Sigma}_{i,j}$. In these cases, the wanted equality is easy to be checked. □

5.2. Some open questions. Let X be a compact Kähler manifold, and let $f : X \to X$ be a dominant meromorphic map.

A) Let T be a positive closed (p,p) current on X with Siu's decomposition $T = R + \sum_j \lambda_j [V_j]$. Let $E(T)$ be as in Theorem 3.7. Assume that for any irreducible analytic V contained in $E(T)$ then $f^{-1}(V)$ has codimension $\geq p$. Is $f^*(T)$ well-defined? If so, is $f^*(R)$ positive? Note that by Corollary 3.7, $f^*(T)$ may not be positive though.

B) Assume that $\pi_1(\mathcal{C}_f)$ has codimension $\geq p$.

a) When $X = \mathbb{P}^k$, [**44**] showed that $\pi_1(\mathcal{C}_{f^n})$ has codimension $\geq p$ for all n. Is the same true for a general X?

b) Does f satisfy dd^c-p stability condition? This holds for $p = 1$.

c) Using a) and the fact that when $X = \mathbb{P}^k$ then f^* preserves the convex cone of positive (p,p) currents, [**44**] showed that if moreover f is p-algebraic stable then $(f^n)^* = (f^*)^n$ for all n. Does the same conclusion hold when X is an arbitrary compact Kähler manifold? Lemma 3.8 shows that the answer to this question is positive when $f = J_X$.

C) Can the constructions of invariant currents in Section 4 be extended to other cases, for example for a map in Question B?

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– Dissertation: Pullback of currents by meromorphic maps
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B.S. Mathematics, University of Science, Ho Chi Minh City, Vietnam 1998-2003
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– Advisor: Dang Duc Trong

Recent publications

Tuyen Trung Truong, *On automorphisms of blowups of* $\mathbb{P}^3$, arXiv:1202.4224.

Tuyen Trung Truong, *Pullback of currents by meromorphic maps*, Bulletin de la Société Mathématique de France, accepted. arXiv: 1107.1743.

Tien-Cuong Dinh, Viet-Anh Nguyen, and Tuyen Trung Truong, *On the dynamical degrees of meromorphic maps preserving a fibration*, Communications in Contemporary Mathematics, accepted. arXiv: 1108.4792.

Tuyen Trung Truong, *Degree complexities of birational maps related to matrix inversions: Symmetric case*, Mathematische Zeitschrift, to appear. arXiv: 1005.4520.

Eric Bedford and Tuyen Trung Truong, *Degree complexities of birational maps related to matrix inversion*, Communications in Mathematical Physics 298 (2010), no. 2, 357-368. arXiv: 0907.3319.

Tuyen Trung Truong, *Sets non-thin at* ∞ *in* $\mathbb{C}^m$, Journal of Mathematical Analysis and Applications 356 (2009), no 2, 517-524. arXiv:0710.2671.

Selected honors and awards

2011-2012: Fernadus and Elizabeth J. Payne fellowship (Dissertation year research fellowship), College of Arts and Sciences, Indiana University Bloomington

2011: William B. Wilcox mathematics award, for outstanding graduate students, Indiana University Bloomington

November 2010: College of Arts and Sciences travel award, Indiana University Bloomington

2008: William B. Wilcox mathematics award, for outstanding graduate students, Indiana University Bloomington

2007: James P. Williams memorial award, for first year outstanding students, Department of Mathematics, Indiana University Bloomington

2007: Robert E. Weber memorial award, for outstanding scholastic achievement on the Tier 1 examination, Department of Mathematics, Indiana University Bloomington

2006-2007: Hazel King Thompson fellowship, Indiana University Bloomington

2001: Third prize for sciencific research (for students) in mathematics, Ministry of Education and Training, Vietnam