

# On the Combinatorics of Schubert Calculus

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## 1 Introduction

We construct a lattice by choosing three unit vectors  $u, v, w$  in the plane such that  $u + v + w = 0$ .

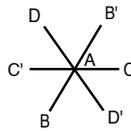


The points  $iu + jv$  with  $i, j$  integers will be called *lattice points*, and a segment joining two nearest lattice points will be called a *small edge*. We consider positive measures  $m$  which are supported by a union of small edges, that satisfy the following properties:

- (1) The restriction of  $m$  to each small edge is a multiple of a linear measure. This multiple is called the *density* of  $m$  on the small edge.
- (2)  $m$  satisfies the balance condition

$$m(AB) - m(AB') = m(AC) - m(AC') = m(AD) - m(AD')$$

whenever  $A$  is a lattice point and the neighboring lattice points  $B, C', D, B', C, D'$  are in cyclic order around  $A$ .



The density of a measure  $m$  will be considered to be zero on segments outside its support. A lattice point incident to at least three small edges in the support of  $m$  is called a *branch point* of the measure  $m$ . We only consider measures with at least one branch point.

Fix an integer  $r \geq 1$ , and denote by  $\Delta_r$  the (closed) triangle with vertices  $0, ru, ru + rv = -rw$ . We use the notation

$$A_j = ju, \quad B_j = ru + jv, \quad C_j = (r - j)w,$$

for  $j = 0, 1, \dots, r$ , for the lattice points on the boundary of  $\Delta_r$ . We also set

$$X_j = A_j + w, \quad Y_j = B_j + u, \quad Z_j = C_j + v$$

for  $j = 0, 1, 2, \dots, r$ .

We denote by  $\mathcal{M}_r$  the collection of all measures  $m$  satisfying conditions (1) and (2), whose branch points are contained in  $\Delta_r$ , and such that

$$m(A_j X_{j+1}) = m(B_j Y_{j+1}) = m(C_j Z_{j+1}) = 0, \quad j = 0, 1, \dots, r.$$

The numbers

$$\alpha_j = m(A_j X_j), \quad \beta_j = m(B_j Y_j), \quad \gamma_j = m(C_j Z_j),$$

where  $j = 0, 1, \dots, r$ , will be called the *exit densities* of  $m$ . A measure  $m \in \mathcal{M}_r$  is said to be rigid if there is no other measure  $m' \in \mathcal{M}_r$  with the same exit points and exit densities as  $m$ . In other words, a rigid measure is entirely determined by its exit densities.

Given a measure  $m \in \mathcal{M}_r$ , we define its weight  $w(m) \in \mathbb{R}_+$  to be

$$w(m) = \sum_{j=0}^r m(A_j X_j) = \sum_{j=0}^r m(B_j Y_j) = \sum_{j=0}^r m(C_j Z_j).$$

The equality of the three sums giving  $w(m)$  is an easy consequence of the balance condition.

The remainder of the paper is organized as follows. In Section 2 we formulate the Littlewood-Richardson Rule in terms of measures. In Section 3 we focus our discussion of measures on a special kind - tree measures. This leads us to our main results in Section 4, where we develop a set of rules for constructing rigid tree measures. We conclude the paper with possible directions for future research in Section 5.

## 2 The Littlewood-Richardson Rule

We can describe the Littlewood-Richardson rule in terms of measures, and this turns out to be a very useful way to study intersections of Schubert varieties. Given integers  $n, 1 \leq k \leq n-1$ , the Grassmanian manifold  $Gr(n, k)$  is defined to be

$$Gr(n, k) = \{k\text{-dimensional linear vector subspaces of } \mathbb{C}^n\}$$

For every flag

$$\mathcal{E} = \{\{0\} = E_0 \subset E_1 \subset E_2 \subset \dots \subset E_n = \mathbb{C}^n\},$$

where  $E_j$  is a subspace of dimension  $j$ ,  $Gr(n, k)$  can be written as a union of Schubert varieties described as follows. For each set

$$I = \{i_1 < i_2 < \dots < i_r\} \subset \{1, 2, \dots, n\},$$

one defines the Schubert variety

$$S(\mathcal{E}, I) = \{M \in Gr(n, k) : \dim(M \cap E_{i_x}) \geq x, x = 1, 2, \dots, k\}.$$

Given sets  $I, J, K \subset \{1, 2, \dots, n\}$  of cardinality  $k$  such that

$$\sum_{l=1}^k (i_l + j_l + k_l - 3l) = 2k(n - k)$$

the Littlewood-Richardson rule provides a non-negative integer  $c_{IJK}$  with the property that the set

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K)$$

has a finite intersection, equal to  $c_{IJK}$ . The integer  $c_{IJK}$  is called the Littlewood-Richardson coefficient, and  $c_{IJK}$  can be defined in terms of measures.

Assume that  $m \in \mathcal{M}_r$  assigns integer densities to all small edges. Let  $\alpha_n, \beta_n, \gamma_n$  be the exit densities of  $m$ . We can then define an integer

$$n = r + w(m),$$

and sets  $I, J, K \subset \{1, 2, \dots, n\}$  of cardinality  $r$  by setting  $I = \{i_1, i_2, \dots, i_r\}$ , where

$$i_l = l + \sum_{n=0}^{l-1} \alpha_n, \quad l = 1, 2, \dots, r, \quad (1)$$

$$j_l = l + \sum_{n=0}^{l-1} \beta_n, \quad l = 1, 2, \dots, r, \quad (2)$$

$$k_l = l + \sum_{n=0}^{l-1} \gamma_n, \quad l = 1, 2, \dots, r. \quad (3)$$

These are precisely the triples of sets  $(I, J, K)$  which satisfy the Littlewood-Richardson rule. The *Littlewood-Richardson coefficient*  $c_{IJK}$  equals the number of measures  $m \in \mathcal{M}_r$  with integer densities which satisfy (1), (2), and (3), i.e. which have the same exit densities as  $m$ .

Given a measure  $m \in \mathcal{M}_r$ , we formulate the associated Schubert intersection problem. A measure  $m$  determines sets  $I, J, K$  as above. The problem is to compute explicit elements in the intersection of three Schubert varieties,

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K),$$

where  $\mathcal{E}, \mathcal{F}, \mathcal{G}$  are generic flag varieties. An explicit solution of the Schubert intersection problem associated with a measure can be produced in the rigid case, and the method is described in [2].

*Example.* In Figure 1, the measure  $m$  assigns density 1 to the thickened edges and density 0 to the other edges in the triangle.

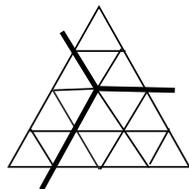


Figure 1: This measure has one branch point.

The sets  $I, J, K$  are determined by  $m$  as follows:

$$\begin{aligned} I &= \{1, 3, 4, 5\} \\ J &= \{1, 3, 4, 5\} \\ K &= \{1, 2, 4, 5\} \end{aligned}$$

The subspace  $M$  of  $\mathbb{C}^5$  in the intersection of three Schubert varieties

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K)$$

is exactly  $M = E_1 + F_1 + G_2$ , where  $E_1, F_1, G_2$  are elements of the flags  $\mathcal{E}, \mathcal{F}, \mathcal{G}$ , respectively.

### 3 Trees and Measures

Some measures  $m \in \mathcal{M}_r$  have an underlying tree structure which we describe next. We consider trees embedded in the plane  $\mathbb{R}^2$  such that

- (T.1) each edge of the tree is a straight line segment of unit length,
- (T.2) each vertex has order 2 or 3, and
- (T.3) there are only finitely many vertices of order 3.

These conditions imply that a tree is infinite, but has a finite number of *ends*. Ends are sequences of vertices of the form  $V_0 V_1 \dots$ , where  $V_0$  is a branch point,  $V_j$  has order 2 for  $j \geq 1$ , and  $V_j V_{j+1}$  is an edge for each  $j \geq 0$ . We will require one more condition on our trees.

- (T.4) The shortest path joining two different ends contains an odd number of vertices of order 3.

All trees discussed in the sequel satisfy the above four properties.

An *immersion* of a tree  $T \subset \mathbb{R}^2$  is a continuous map  $f : T \rightarrow \mathbb{R}^2$  which satisfies the following properties:

- (1)  $f$  is isometric on each edge.

(2) If  $VA$  and  $VB$  are two edges meeting at a vertex of order 2, then

$$2f(V) = f(A) + f(B).$$

(3) If  $VA, VB, VC$  are three edges meeting at a vertex of order 3, then

$$3f(V) = f(A) + f(B) + f(C),$$

and the restriction of  $f$  to  $VA \cup VB \cup VC$  preserves the orientation of the tree.

A tree is endowed with an arclength measure. Given an immersion  $f$  of  $T$ , we consider the push-forward  $m_f$  of this measure. That is, if  $f(T)$  is contained in the small edges of a lattice determined by the vectors  $u, v, w$ , then  $m_f$  assigns to each edge a density equal to the number of its preimages in  $T$ . The resulting measure satisfies the balance condition at all vertices. Since  $T$  has a finite number of ends, we can arrange  $f$  so that  $m_f$  belongs to  $\mathcal{M}_r$  for sufficiently large  $r$ .

A measure  $m$  will be called a *tree measure* if  $m = m_f$  for some immersion  $f$  of a tree. In the next section, we will construct a set of rules for which an immersion produces a rigid tree measure.

## 4 Results

There is a certain class of loops which indicate non-rigidity if they exist in the support of a measure. Let  $A_1 A_2 \dots A_k A_1$  be a loop consisting of small edges  $A_j A_{j+1}$  contained in the support of a measure  $m \in \mathcal{M}_r$ . We will say that this loop is *evil* if each three consecutive points  $A_{j-1} A_j A_{j+1} = ABC$  forms an *evil turn*, i.e. one of the following situations occurs:

- (E.1)  $C = A$ , and the small edges  $BX, BY, BZ$  which are  $120^\circ, 180^\circ$ , and  $240^\circ$  clockwise from  $AB$  are in the support of  $m$ .
- (E.2)  $BC$  is  $120^\circ$  clockwise from  $AB$ .
- (E.3)  $C \neq A$  and  $A, B, C$  are collinear.
- (E.4)  $BC$  is  $120^\circ$  counterclockwise from  $AB$  and the edge  $BX$  which is  $120^\circ$  clockwise from  $AB$  is in the support of  $m$ .
- (E.5)  $BC$  is  $60^\circ$  counterclockwise from  $AB$  and the edges  $BX, BY$  which are  $120^\circ$  and  $180^\circ$  clockwise from  $AB$  are in the support of  $m$ .

The existence of an evil loop in the support of a measure implies non-rigidity. This was proven in [1]:

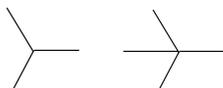
**Theorem 4.1.** *A measure  $m \in \mathcal{M}_r$  is rigid if and only if its support contains none of the following configurations:*

- (1) Six edges meeting at one lattice point.
- (2) An evil loop.

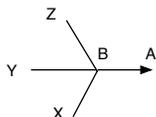
Our main result is to prove the following proposition:

**Proposition 4.2.** *Let  $T$  be a tree that satisfies properties (T.1) through (T.4), and  $f$  an immersion  $f : T \rightarrow \mathbb{R}^2$ . Suppose  $f$  satisfies the following conditions:*

- (1) *There is a vertex  $A \in T$  (called the root of  $T$ ) such that  $f^{-1}(f(A)) = \{A\}$ .*
- (2) *The only branch points of  $f(T)$  are of the following forms (up to rotation):*



- (3)  *$f$  has consistent orientation. That is, suppose  $X_1X_2$  and  $Y_1Y_2$  are edges of  $T$  such that  $X_1$  and  $Y_1$  are closer to the root of  $T$  than  $X_2$  and  $Y_2$ , respectively, and  $f(X_1X_2) = f(Y_1Y_2)$ . Then  $f(X_1) = f(Y_1)$  (and consequently,  $f(X_2) = f(Y_2)$ ).*
- (4) *If four edges meet at a lattice point  $B$ , then the orientation of one of the edges is determined as follows. Let  $AB$  be the small edge such that the other small edges  $BX, BY, BZ$  are located  $120^\circ, 180^\circ, 240^\circ$  clockwise from  $AB$ . Then the orientation of  $AB$  must point from  $B$  to  $A$ .*



Then  $m_f$  is a rigid tree measure.

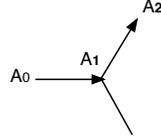
*Proof.* Suppose  $f : T \rightarrow \mathbb{R}^2$  satisfies the conditions of the hypothesis. By Rule (2), a branch point in  $f(T)$  cannot have six surrounding edges all belonging to  $f(T)$ . By Theorem 4.1, it is enough to show that  $f(T)$  contains no evil loops. We show that if an evil loop exists in  $f(T)$ , then the loop lifts to a single branch of  $T$ . This will lead to a contradiction of property T.3.

By the conditions imposed on the immersion  $f$ , the only evil turns that can arise in  $f(T)$  are E.1, E.2, E.3 and E.4. The evil turn E.5 cannot arise because of Rule (2). The turns E.2 and E.4 are reversible under our conditions, in the sense that they are evil either way we traverse them.

Let  $A_0A_1A_2 \cdots A_n$  be an evil loop in  $f(T)$  (where  $A_0 = A_n$ ). Since an evil loop must have a turn of the form E.2, E.4, or E.5 (because collinear "turns" alone cannot form a loop), we may choose  $A_0$  so that the evil turn  $A_0A_1A_2$  is of the form E.2 or E.4. Since the turns E.2 and E.4 are reversible, we may assume that  $A_0A_1$  assumes the orientation of the tree (if not, we traverse the evil loop

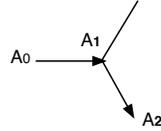
in the reverse order). That is, we can assume that there is a lift  $B_0B_1$  to the tree of the edge  $A_0A_1$ . We show that  $A_1A_2$  can also be lifted by showing that the orientation of  $A_1A_2$  matches the tree orientation. The possible configurations of  $A_0A_1A_2$  are:

- (E.2)  $A_1A_2$  is  $120^\circ$  clockwise from  $A_0A_1$ .  
 The edge  $A_1X$  which is  $240^\circ$  clockwise from  $A_0A_1$  must also be in the support of  $f(T)$ .



The edge  $A_1A_2$  must assume the orientation of the tree by consistency of orientation of the immersion  $f$ .

- (E.4)  $A_1A_2$  is  $120^\circ$  counterclockwise from  $A_0A_1$  and the edge  $A_1X$  which is  $120^\circ$  clockwise from  $A_0A_1$  is in the support of  $f(T)$ .



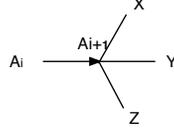
The edge  $A_1A_2$  must assume the orientation of the tree by consistency of orientation of the immersion  $f$ .

Thus, given a lift  $B_0B_1$  of  $A_0A_1$ , there is a vertex  $B_2$  such that  $B_1B_2$  lifts  $A_1A_2$ . So the evil turn  $A_0A_1A_2$  lifts to a simple path  $B_0B_1B_2$  on  $T$ , consistent with the orientation of  $T$ .

We inductively show that each edge in the evil loop  $A_0 \dots A_n$  lifts to an edge of the tree. For simplicity, we let  $A_{k+mn} = A_k$  for all  $m \in \mathbb{N}$ ,  $0 \leq k < n$ .

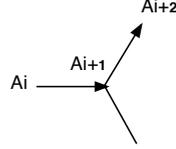
Suppose there is a lift  $B_iB_{i+1}$  of the edge  $A_iA_{i+1}$  so that  $A_iA_{i+1}$  assumes the orientation of the tree. We show that there is also a lift  $B_{i+1}B_{i+2}$  of the edge  $A_{i+1}A_{i+2}$ . The possible configurations of the evil turn  $A_iA_{i+1}A_{i+2}$  are:

- (E.1)  $A_{i+2} = A_i$ , and the small edges  $A_{i+1}X$ ,  $A_{i+1}Y$ ,  $A_{i+1}Z$  which are  $120^\circ$ ,  $180^\circ$ , and  $240^\circ$  clockwise from  $A_iA_{i+1}$  are in the support of  $f(T)$ .



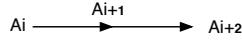
The orientation of the edge  $A_iA_{i+1}$  in this evil turn contradicts Rule (4) of the immersion  $f$ , so this evil turn does not arise in  $f(T)$ .

- (E.2)  $A_{i+1}A_{i+2}$  is  $120^\circ$  clockwise from  $A_iA_{i+1}$ . The edge  $A_{i+1}X$  which is  $240^\circ$  clockwise from  $A_iA_{i+1}$  must also be in the support of  $f(T)$ .



The edge  $A_{i+1}A_{i+2}$  must assume the orientation of the tree by consistency of orientation of the immersion  $f$ .

- (E.3)  $A_{i+2} \neq A_i$  and  $A_i, A_{i+1}, A_{i+2}$  are collinear. There are two cases:
- The lattice point  $A_{i+1}$  has only two surrounding edges,  $A_iA_{i+1}$  and  $A_{i+1}A_{i+2}$ , in the support of  $f(T)$ .



Then the edge  $A_{i+1}A_{i+2}$  must assume the orientation of the tree by consistency of orientation of the immersion  $f$ .

- The lattice point  $A_{i+1}$  has four surrounding edges in the support of  $f(T)$ .

By Rule (4), the edge  $A_{i+1}A_{i+2}$  must assume the orientation of the tree.

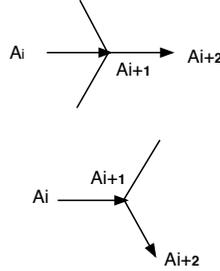
- (E.4)  $A_{i+1}A_{i+2}$  is  $120^\circ$  counterclockwise from  $A_iA_{i+1}$  and the edge  $A_{i+1}X$  which is  $120^\circ$  clockwise from  $A_iA_{i+1}$  is in the support of  $f(T)$ .

The edge  $A_{i+1}A_{i+2}$  must assume the orientation of the tree by consistency of orientation of the immersion  $f$ .

Thus, given a lift  $B_iB_{i+1}$  of  $A_iA_{i+1}$ , there is a vertex  $B_{i+2}$  such that  $B_{i+1}B_{i+2}$  lifts  $A_{i+1}A_{i+2}$ , completing the induction.

We have constructed inductively an infinite path

$$B_0B_1B_2 \cdots B_nB_{n+1} \cdots$$



of the tree  $T$  with the following properties:

- (1)  $f(B_j) = A_j$
- (2)  $B_j B_{j+1}$  is oriented away from the root.

We claim that infinitely many of the vertices  $B_j$  have order 3.

By construction, the vertex  $A_1$  is a branch point of  $m_f$ . Thus lifted vertex  $B_1 \in T$  with  $f(B_1) = A_1$  is a vertex of order 3, because the immersion  $f$  cannot map a vertex of order 2 to a turn of the form E.2 or E.4.

Moreover, each vertex  $B_{1+mn}$ ,  $m \in \mathbb{N}$ , has order 3 because

$$f(B_{1+mn}) = A_{1+mn} = A_1.$$

Thus, infinitely many vertices  $B_j$  have order 3. But this contradicts property T.3 of the tree. Thus, we've shown that an evil loop cannot exist in  $f(T)$ , and therefore  $m_f$  is a rigid tree measure.  $\square$

The above proposition guarantees that if we construct a tree measure following the four stated rules, then the resulting measure will be rigid.

## 5 Future Research

The finiteness of the number of vertices of order 3 (i.e. property T.3) was crucial in the proof of Proposition 4.2. Our next question is whether Proposition 4.2 still applies to trees with an infinite number of vertices of order 3. That is, if a tree satisfies properties T.1, T.2, and T.4, and if an immersion  $f : T \rightarrow \mathbb{R}^2$  satisfies the four rules of Proposition 4.2, then will the resulting measure  $f(T)$  be rigid?

Another question we investigated is the following: Given a triangle of size  $r$ , what is the maximum weight that a rigid tree measure  $m \in \mathcal{M}_r$  could have? We have a lower bound on this number: for large  $r$ , the maximum weight of a measure in  $\mathcal{M}_r$  is at least  $2^{\lfloor r/3 \rfloor}$ . The lower bound can be easily explained with the help of a couple of figures.

Given a measure  $m \in \mathcal{M}_r$  of weight  $w$ , Figures 2 and 3 demonstrate how to produce a measure  $m' \in \mathcal{M}_{r+3}$  of weight  $2w$ . By branching the exit densities on one side of  $\Delta_r$  in the illustrated way, one exit branch of density  $w$  is added

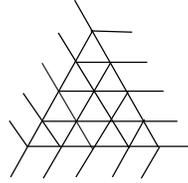


Figure 2: A measure  $m \in \mathcal{M}_4$  of weight  $w$

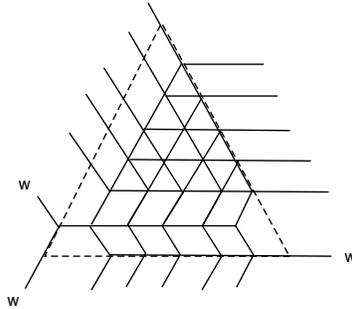


Figure 3: A measure  $m' \in \mathcal{M}_7$  of weight  $2w$

to each side of the triangle, thereby increasing the total weight by a factor of 2. In this process, we also increase the size of the triangle by 3. Thus, for large  $r$ , the maximum weight of  $m \in \mathcal{M}_r$  is  $\geq 2^{\lfloor r/3 \rfloor}$ . We are interested in finding a least upper bound on the maximum weight of a measure.

A tree measure of weight  $w$  has  $3w - 2$  branch points. This relation allows us to investigate the maximal weight problem using a different approach. The problem becomes that of finding the maximum number of branch points of a rigid tree measure in  $\mathcal{M}_r$ .

Thus, we attempted to characterize trees with rigid immersions. That is, we are interested in the types of trees that can be immersed onto the plane to produce rigid measures. Does every tree have a rigid immersion? Are there some tree configurations that absolutely prohibit a rigid immersion? By characterizing trees with rigid immersions, we might be able to find an upper bound on the number of branch points that a rigid tree measure  $m \in \mathcal{M}_r$  can have, and ultimately find the maximum weight that such a measure could have.

## 6 Acknowledgements

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## References

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