Torsion Subgroups of CAT(0) Groups

Nathan Dowlin *Hamza Ghadyali*

Abstract

Given a CAT(0) group G acting geometrically on a proper CAT(0) space, we attempt to demonstrate that any torsion subgroup of G has finite cardinality.

1 Introduction

Euclid's fifth postulate, commonly referred to as the parallel postulate, states that given a line ℓ and a point p not on ℓ there is exactly one line through p that does not intersect ℓ. When this postulate is removed, we allow for a much larger class of metric spaces which can have peculiar properties. For example, in hyperbolic geometry there are infinitely many lines through p that do not intersect ℓ. Furthermore, in elliptic geometry no such line exists since every line through p will intersect ℓ. Indeed, these are salient features of $\mathbb{H}^n$, hyperbolic n-space, and $\mathbb{S}^n$, the n-sphere, respectively. Together, $\mathbb{H}^n$, $\mathbb{S}^n$, and $\mathbb{E}^n$ (Euclidean n-space), are the model spaces of constant negative, positive, and 0 curvature, respectively.

CAT(0) spaces are of particular interest because while they are incredibly general, the lack of positive curvature provides us with many powerful tools. In this paper, we are primarily concerned with CAT(0) groups, or groups which act geometrically on proper CAT(0) spaces. More precisely, a CAT(0) group is a subgroup of the group of isometries of a proper CAT(0) space that acts properly discontinuously and cocompactly on the space. In studying these groups, one natural question that arises is whether there can exist an infinite subgroup consisting entirely of torsion (finite order) elements. Not only is this question interesting in its own right, but an important cut point theorem by Eric Swenson requires this to be true in order to complete his proof. For $\mathbb{E}^n$, Bieberbach has shown that no such infinite torsion subgroup can exist, and the same result has been shown for δ-hyperbolic groups; we studied Olshanskii's proof of this theorem in great depth. CAT(0) spaces lie somewhere in between these two, and a similar result is expected for CAT(0) groups.

We use various geometric approaches to arrive at a proof of this result. At first, since the CAT(0) case most closely resembles the situation in δ-hyperbolic groups, we attempted to follow Olshanskii's proof substituting CAT(0) groups for δ-hyperbolic groups. However, without word-hyperbolicity, word-processing in the CAT(0) group became incredibly complicated very quickly. Instead, we resorted to the geometry of CAT(0) spaces and examined how it must restrict the groups acting on them. Because Swenson's proof requires only that torsion subgroups fixing a point in the boundary at infinity to be finite, we assume that our torsion subgroups do in fact have a fixed point in the boundary. Using this assumption, we have derived many characteristics regarding the manner in which torsion groups act on horospheres. While we have been unable to prove the theorem in general, these characteristics have allowed us to show something weaker about a torsion group T fixing a point c in the boundary at infinity. We proved that if the boundary at infinity of the horosphere at c consists only of the point c, the torsion subgroup can only be infinite if there exists an infinite torsion group with a finite number of conjugacy classes, which is currently a major open question in group theory.

2 Background

We provide the basic definitions that recur throughout this paper here.

Definition 2.1. Let (X, d_x) and (Y, d_y) be a metric spaces. An *isometry* between X and Y is a map $\phi\colon X \to Y$ such that $\forall x, x' \in X$, $d_x(x, x') = d_y(\phi(x), \phi(x'))$. A *(unit speed) geodesic* in X is an isometry $\gamma\colon I \to X$ where I is a connected subset of $\mathbb{R}$. If $I = [a, b]$, $I = [a, \infty)$, or $I = \mathbb{R}$, then γ is a *geodesic segment, ray, or line*, respectively.

Definition 2.2. Let X be a metric space. We call X a *geodesic metric space* if $\forall x, x' \in X$ there is a geodesic segment between x and x', (ie $\exists\gamma\colon [0, a] \to X$ such that $\gamma(0) = x$ and $\gamma(a) = x'$). If $\forall x, x' \in X$ this geodesic segment is unique, X is a *unique geodesic metric space.*

The definition below for a CAT(0) space comes from [Ru].

Definition 2.3. Let (X, d) be a metric space. X is *proper* if closed metric balls are compact. Let (X, d) be a proper complete geodesic metric space. If $\triangle abc$ is a geodesic triangle in X, then we consider $\triangle\bar{a}\bar{b}\bar{c}$ in $\mathbb{E}^2$, a triangle with the same side lengths, and call this a *comparison triangle.* Let $\triangle abc$ be a geodesic triangle in X. Then $\triangle abc$ satisfies the CAT(0) *inequality* if for any comparison triangle and any two points p, q on $\triangle abc$, the corresponding points $\bar{p}, \bar{q}$ on the comparison triangle satisfy

$$d(p, q) \leq d_{\mathbb{E}^2}(\bar{p}, \bar{q})$$

If every geodesic triangle in X satisfies the CAT(0) inequality, then we say X is a CAT(0) space.

Remark 2.4. If (X, d) is a CAT(0) space, then the following hold:

(a) The distance function $d\colon X \times X \to \mathbb{R}$ is convex.

(b) X is a unique geodesic metric space.

(c) X is contractible.

Definition 2.5. G is a CAT(0) group if there exists a CAT(0) space X such that G is a subgroup of the group of isometries of X and G acts both properly discontinuously and cocompactly on X. (G acts cocompactly if the fundamental domain is compact.)

Example 2.6. A simple example of a CAT(0) group is $\mathbb{Z}^2$ with the action on $\mathbb{R}^2$ by translations.

Definition 2.7. Let G be a CAT(0) group, and let L be a proper subgroup. Let S be a set of generators such that $< S >= L$. If $g \in G \setminus L$, then the group $< S \cup g >$ will be called the *extension* of L by g. For notational convenience, this group will be written as $Ext(L, g)$.

Definition 2.8. Let G be a CAT(0) group, and let T be a torsion subgroup. We say that T is a *maximal torsion subgroup* of G if $\forall g \in G \setminus T$, the group $Ext(T, g)$ is not a torsion group (i.e. it contains a translation).

Definition 2.9. Two geodesic rays $\gamma, \gamma' \colon [0, \infty) \to X$ are said to be *asymptotic* if there exists a constant K such that $d(\gamma(t), \gamma'(t)) \leq K$ for all $t \geq 0$. This gives an equivalence relation. The set of all equivalence classes forms the *boundary at infinity of* X, denoted $\partial_\infty X$. If γ is in the equivalence class c, we say γ *goes to* c or γ *approaches* c.

The action of an isometry on X has a natural extension to $\partial_\infty X$. Let $g \in Isom(X)$, where $Isom(X)$ is the group of all isometries from the metric space X to itself. Given two geodesic rays $\gamma, \gamma' \colon [0, \infty) \to X$, if there exists K such that $d(\gamma(t), \gamma'(t)) \leq K$ for all $t \geq 0$, then $d(g(\gamma(t)), g(\gamma'(t))) \leq K$ for all $t \geq 0$. Therefore, $g \circ \gamma$ and $g \circ \gamma'$ are in the same equivalence class, so g preserves equivalence classes of geodesic rays and hence extends to a map on the boundary.

Example 2.10. It is easy to see that both $\partial_\infty \mathbb{E}^2$ and $\partial_\infty \mathbb{H}^2$ is $\mathbb{S}^1$.

Definition 2.11. Let $B(p, r)$ denote the ball centered at p of radius r. Let $\gamma \colon [0, \infty) \to X$ with $\gamma(0) = x_0$ be a geodesic ray emanating from x_0 in the equivalence class c. We define the *horoball at* c *determined by* γ to be

$$\mathcal{B}_{c,\gamma} = \bigcup_{t \in [0,\infty)} B(\gamma(t), d(\gamma(t), x_0))$$

Since $\gamma(0) = x_0$ and γ a geodesic, $d(\gamma(t), x_0)) = t$. The definition can then be written more concisely as

$$\mathcal{B}_{c,\gamma} = \bigcup_{t \in [0,\infty)} B(\gamma(t), t)$$

The corresponding *horosphere at* c *determined by* γ denoted by $\mathcal{H}_{c,\gamma}$ is the boundary of the horoball $\mathcal{B}_{c,\gamma}$. With this notation, the geodesic ray γ is required to be in the equivalence class c. When the context is clear or we are referring to any horosphere at c, we will drop the second subscript.

Definition 2.12. Equivalently, we can define the horosphere using the Busemann function. Let γ be a geodesic ray defined as in the previous definition. Given a point $y \in X$ we define the *Busemann function* with respect to γ

$$b_\gamma(y) = \lim_{t\to\infty}(d(\gamma(t), y) - t)$$

The horosphere is then the level set $\mathcal{H}_{c,\gamma} = b_\gamma^{-1}(0)$, and similarly the horoball is $\mathcal{B}_{c,\gamma} = b_\gamma^{-1}((-\infty, 0])$.

Definition 2.13. We extend the notion of boundary at infinity to horoballs and horospheres as follows. If $\mathcal{B}_c$ is a horoball at c and $\mathcal{H}_c$ is the corresponding horosphere, then we will denote the boundary at infinity of these sets by $\partial_\infty \mathcal{B}_c$ and $\partial_\infty \mathcal{H}_c$ respectively.

$\partial_\infty \mathcal{B}_c$ is defined to be the set of equivalence classes of all geodesic rays γ which, if $\gamma(0) \in \mathcal{B}_c$, then $\gamma(t) \in \mathcal{B}_c$ for all $t \in [0, \infty)$. $\partial_\infty \mathcal{H}_c$ is naturally the boundary of $\partial_\infty \mathcal{B}_c$ as a subset of $\partial_\infty X$.

3 Results

The main theorems we use regarding properties of CAT(0) groups and their torsion subgroups can be found in [BH]. These results will be stated without proof. Throughout this section, G refers to a CAT(0) group acting on a CAT(0) space X properly discontinuously and cocompactly, and T denotes a subgroup of G consisting of only torsion elements. We will make the assumption throughout that T fixes a point in the boundary at infinity, and we will call this point c.

Lemma 3.1. *Every torsion element t fixes a point $p_t \in X$*

Proof. Let $t \in G$ be a torsion element and define $F =< t >$. Let $x \in X$ and let Fx be the orbit of x under F. Since F is finite, this is a bounded set. Therefore, $\mathcal{C}(Fx)$ is a compact, convex set, where $\mathcal{C}(Fx)$ denotes the convex hull of the set Fx. Moreover, $t(Fx) = Fx$, so $t(\mathcal{C}(Fx)) = \mathcal{C}(Fx)$. Applying the Schauder Fixed Point Theorem, t fixes a point in $\mathcal{C}(Fx)$. □

Remark 3.2. Since each torsion element t fixes a point $p_t \in X$, it also fixes the geodesic ray emanating from p_t in the equivalence class c.

Lemma 3.3. *CAT(0) groups have finitely many conjugacy classes of finite subgroups.*

Lemma 3.3 implies directly that CAT(0) groups have finitely many conjugacy classes of finite order elements. Therefore, given a torsion group $T \subseteq G$, the equivalence relation $t_1 \sim t_2$ if $\exists g \in G$ such that $t_1 = g t_2 g^{-1}$ divides T into a finite number of equivalence classes. We will call these equivalence classes *conjugacy classes of T over G.*

Lemma 3.4. *Let $\mathcal{H}_{c,\gamma}$ be a horosphere at c. If $g \in T$, then g stabilizes $\mathcal{H}_{c,\gamma}$.*

Proof. Let $\mathcal{H}_{c,\gamma}$ be the horosphere at c determined by the geodesic ray γ. Define $x \in \mathcal{H}_{c,\gamma}$ by $x = \gamma(0)$. Let $g \in T$, and let $y \in X$ be a fixed point for g. Then g fixes α, where α is the geodesic ray from y to c. Let $a \in \mathcal{H}_{c,\alpha}$.

$$b_\alpha(ga) = \lim_{t\to\infty}(d(\alpha(t), ga) - t)$$

Since g^{-1} is an isometry,

$$d(\alpha(t), ga) = (d(g^{-1}(\alpha(t)), a)$$

and

$$b_\alpha(ga) = \lim_{t\to\infty}(d(g^{-1}(\alpha(t)), a) - t)$$

But for all t, $g^{-1}(\alpha(t)) = \alpha(t)$, so $b_\alpha(ga) = \lim_{t\to\infty}(d(\alpha(t), a) - t) = b_\alpha(a) = 0$. This is true for all a in $\mathcal{H}'_c$, so $g(\mathcal{H}_{c,\alpha}) = \mathcal{H}_{c,\alpha}$. The geodesic ray α intersects all horospheres inside $\mathcal{H}_{c,\alpha}$, and this intersection is fixed by g, so the same argument applies to all horospheres inside $\mathcal{H}'_c$. Therefore, if $\mathcal{H}_c \subset \mathcal{B}_{c,\alpha}$, then $\mathcal{H}_{c,\gamma}$ is stabilized by g.

Assume $\mathcal{H}_{c,\alpha}$ lies outside $\mathcal{H}_{c,\alpha}$. Then γ intersects $\mathcal{H}_{c,\alpha}$; let $z = \gamma(t_0)$ be the point at which they intersect. Note that $d(x, z) = t_0$, so $\mathcal{H}_{c,\gamma}$ is the unique horosphere a distance t_0 outside of $\mathcal{H}_{c,\alpha}$.

Let $b \in \mathcal{H}_c$. As before,

$$b_\gamma(gb) = \lim_{t\to\infty}(d(\gamma(t), gb) - t) = \lim_{t\to\infty}(d(g^{-1}(\gamma(t)), b) - t)$$

Since g^{-1} fixes c, $g^{-1}(\gamma)$ is also a geodesic ray going to c. Therefore, g maps $\mathcal{H}_c$ to another horosphere at c. Additionally, we have

$$d(x, z) = t_0 = d(g^{-1}(x), g^{-1}(z))$$

and g^{-1} stabilizes $\mathcal{H}'_c$, so $g^{-1}(z)) \in \mathcal{H}'_c$. Combining this with the fact that $g^{-1}(x)$ and $g^{-1}(z)$ lie on a geodesic ray going to c, we see that $g^{-1}(\mathcal{H}_c)$ is either the horosphere t_0 inside $\mathcal{H}'_c$ or the horosphere t_0 outside $\mathcal{H}'_c$. But if $g^{-1}(\mathcal{H}_c)$ is inside $\mathcal{H}'_c$, then so is $g(g^{-1}(\mathcal{H}_c))$, which contradicts our assumption that $\mathcal{H}_c$ lies outside $\mathcal{H}'_c$. $g^{-1}(\mathcal{H}_c)$ must then be the horosphere t_0 outside $\mathcal{H}'_c$, which is $\mathcal{H}_c$. Hence, g stabilizes the horosphere $\mathcal{H}_c$.

□

Lemma 3.5. *Given a horosphere $\mathcal{H}_c$, if there exists a geodesic line γ such that $\gamma \subseteq H_c$, then $\partial_\infty \mathcal{H}_c$ contains at least two points.*

Proof. Define $\alpha, \beta\colon [0,\infty) \to X$ by $\alpha(t) = \gamma(t)$ and $\beta(t) = \gamma(-t)$. Then α and β are geodesic rays in $\mathcal{H}_c$. Assume $\partial_\infty \mathcal{H}_c$ consists of only c. Then α and β are geodesic rays from $\gamma(0)$ to c, so by uniqueness of geodesics, $\alpha = \beta$. But if $\gamma(-1) = \gamma(1)$, then the geodesic segment $\gamma([-1, 1])$ is a loop, which is a contradiction. Therefore, $\partial_\infty \mathcal{H}_c$ contains at least two points. □

Lemma 3.6. *Let $x \in X$ be a translation, and let $a, b \in \partial_\infty X$ be endpoints of the axis of x. If there exists $c \in \partial_\infty X$ such that $xc = c$ and $a \neq c \neq b$, then the axis of x is contained in a horosphere at c.*

Proof. Let $\gamma\colon [0,\infty) \to X$ denote the axis of h, and let r be the distance that h translates points on the axis. Note that since γ is the axis of a translation, it is also a geodesic line. Assume γ is not contained in any horosphere at c.

Let $\mathcal{H}_c$ be a horosphere at c such that $\mathcal{H}_c \cap \gamma$ consists of exactly two points at least r apart (we can do this because horospheres are convex; we just have to choose one large enough). Call these two points u and v, with

$$d(u,v) = d(xu,v) + r$$

Then hv and $h^{-1}u$ are inside the horosphere $\mathcal{H}_c$ and hu and $h^{-1}v$ lie outside $\mathcal{H}_c$.

Let $\alpha\colon [0,\infty) \to X$ be the geodesic ray from u to c. Then $\mathcal{H}_c = b_\alpha^{-1}(0)$. h fixes c, so $h(\alpha)$ and $h^{-1}(\alpha)$ are geodesics going to c. Let $x \in \mathcal{H}_c$. Then

$$b_\alpha(hx) = \lim_{t\to\infty}(d(\alpha(t), hx) - t) = \lim_{t\to\infty}(d(h^{-1}(\alpha(t)), x) - t) = b_{h^{-1}\alpha}(x)$$

This is the same for any x in $\mathcal{H}_c$, so h maps $\mathcal{H}_c$ to another horosphere at c. Call this horosphere $\mathcal{H}'_c$.

So we have that $h(\mathcal{H}_c) = \mathcal{H}'_c$. But $h(u)$ lies outside the horosphere $\mathcal{H}_c$ and $h(v)$ lies inside, so they can not lie on the same horosphere at c, so we have reached a contradiction. Therefore, γ is contained in some horosphere at c. □

Proposition 3.7. *Let $\mathcal{H}_c$ be a horosphere at c, where $c \in \partial_\infty X$. If there exists $c' \in \partial_\infty X$ and a horosphere $\mathcal{H}_{c'}$ at c' such that $c \in \partial_\infty \mathcal{H}_{c'}$, then $c' \in \partial_\infty \mathcal{H}_c$.*

Lemma 3.8. *Let C be an infinite conjugacy class of T (over G), and let $m \in C$. Then $C = \{x_i m x_i^{-1}\}_{i=1}^\infty$, where $x_i \in G$. Suppose each x_i a translation. Let $a_i \in \partial_\infty X$ be the point farthest from c which is fixed by $x_i m x_i^{-1}$ and let c_i be the point at infinity towards which x_i translates. If $\lim_{i\to\infty} x_i c = \lim_{i\to\infty} a_i = c$ and $\lim_{i\to\infty} c_i = c$, then $\lim_{i\to\infty} |< x_i m x_i^{-1} m^{-1} >| = \infty$. In particular, if x is a translation towards c, then $|< xmx^{-1}m^{-1} >| = \infty$.*

Proof. Sketch of proof.

We will begin with the particular case, when x is a translation towards c. Let $\gamma\colon (-\infty,\infty) \to X$ be the axis of x oriented such that $\gamma(\infty) = c$, and r be the distance that x maps points on the axis. Define $\alpha\colon [0,\infty) \to X$ by $\alpha(t) = \gamma(t)$. Then x maps the horosphere $\mathcal{H}_{c,\alpha}$ to the horosphere $\mathcal{H}_{c,\beta}$, where $\beta(t) = \gamma(t+r)$. Geometrically, x maps a horosphere at c to the unique horosphere a distance r inside it.

Now examine xmx^{-1}. Since x^{-1} maps horospheres at c *outward* by r, it acts as a contraction on geodesics going to c centered at γ, so it can be thought of as contracting the horosphere as it moves it outwards. The torsion element m acts as a rotation or reflection about some fixed set in the horosphere, so the contraction has very little effect on the action of m except that it can translate this fixed set, so m is rotating or reflecting about a different point or set. x then maps this horosphere back to the original horosphere by a dilation about γ. The element xmx^{-1} therefore acts in the same way as m, but centered at a different point on the horosphere. We believe that this implies $xmx^{-1}m^{-1}$ is a translation, and we get our intuition from the manifold case. A simple example is rotations in the Euclidean plane: if a is a rotation by θ centered at p and b is a rotation by θ centered at q, then ab^{-1} is a translation.

In the more more general part of our lemma, we have an infinite conjugacy class $C = \{x_i m x_i^{-1}\}_{i=1}^{\infty}$. Let x_i and t_i have the properties assumed in the lemma. Then the sequence $\lim_{i\to\infty} x_i m x_i^{-1} m^{-1}$ is limiting to the above case, so it is limiting to a translation. Therefore, the order must approach infinity. If we were to compare this to Euclidean case mentioned previously, this sequence is analogous to a sequence ab_i^{-1}. As before, a is a rotation by θ centered at p, and we define b_i to be a rotation by θ_i centered at qi. If $\lim_{i\to\infty} \theta_i = \theta$ and if $\lim_{i\to\infty} q_i = q$, then the element ab_i^{-1} is a rotation by an angle ϕ_i, with $\lim_{i\to\infty} \phi_i = 0$. Thus, $\lim_{i\to\infty} | < ab_i^{-1} > | = \infty$.

$\square$

Theorem 3.9 (Main Theorem). *Let T be a maximal torsion subgroup of G, and let $c \in \partial_\infty X$ be fixed by T. If $\partial_\infty H_c = \{c\}$, then T has a finite number of conjugacy classes.*

Proof. We have that $\forall t \in T$, $tc = c$. Let $W \subseteq T$ be the subset consisting of all elements that fix at least one other point in the boundary (i.e. $t \in W$ if $t \in T$ and $\exists c_t \in \partial_\infty X$ such that $c_t \neq c$ and $tc_t = c_t$).

Assume W is a finite set. Recall that G has a finite number of conjugacy classes of finite order elements, so T has a finite number of conjugacy classes over G. Let C be one such conjugacy class, and let $m \in C$. Each $t \in C$ can then be written as xmx^{-1} for some $x \in G$. But $t(xc) = xmx^{-1}(xc) = xm(c)$. Since $m \in T$, $mc = c$, and $t(xc) = xc$. Therefore, xc is a fixed point for t. We assumed that W is finite, so this tells us that if $G_c \subset G$ is the stabilizer subgroup of G fixing c, then T has a finite number of conjugacy classes over G_c. Let $C_1, C_2, ...C_n$ be the conjugacy classes of T over G_c, and let $m_i \in C_i$. Each element $t \in T$ can be written xm_jx^{-1}, where $t \in C_j$ and $x \in G_c$.

Case 1: $\forall t \in T, \exists x \in G_c$ such that x is torsion and $t = xm_jx^{-1}$ for some j. All such x's fix c, and T is maximal, so either every such x is an element of T, or $\exists x$ such that the group extension of T by x is not a torsion group.

In the latter case, there must be some element $z \in Ext(T, x)$ such that $| < z > | = \infty$. But T and x both stabilize horospheres at c, so z must as well. Let a and b be the endpoints of the axis of z. Let $\mathcal{H}_c$ be a horosphere at c, and let $p \in \mathcal{H}_c$. Then $\forall n \in \mathbb{Z}, z^n(p) \in \mathcal{H}_c$, so $\lim_{n\to\infty} z^n(p) = a \in \partial_\infty \mathcal{H}_c$ and $\lim_{n\to-\infty} z^n(p) = b \in \partial_\infty \mathcal{H}_c$. Since $a \neq b$, this violates $\partial_\infty \mathcal{H}_c = \{c\}$.

Therefore, every such x is an element of T, so T has a finite number of conjugacy classes.

Case 2: There exists $t \in C_j$ such that $t = xm_jx^{-1}$ and x a translation fixing c. If c is an endpoint for the axis of x, then by Lemma 3.8 $| < tm^{-1} > | = \infty$, which violates T being a torsion group. If c is not an endpoint for the axis of x, then by Lemma 3.6 the axis of x is contained in a horosphere at c. The axis is a geodesic, so Lemma 3.5 tells us that $\partial_\infty \mathcal{H}_c$ contains at least two points, so $\partial_\infty \mathcal{H}_c \neq \{c\}$.

Therefore, whenever W is finite, T has a finite number of conjugacy classes.

Assume now that W is infinite. Since T has a finite number of conjugacy classes over G, at least one of these conjugacy classes must contain an infinite subset of W. Let C be one such conjugacy class, and let $m \in W \cap C$. Let $S \subset \partial_\infty X$ be the set of points at infinity fixed by m. Since $m \in W$, S consists of at least two points.

Each element $t \in C$ can be written as xmx^{-1} for some x in G. Let $a \in S$.

$$t(xa) = (xmx^{-1})(xa) = xm(a) = xa$$

Therefore, if $t = xmx^{-1}$, then t fixes the set xS. We will now enumerate elements of C as $\{x_i m x_i^{-1}\}_{i=1}^{\infty}$, and define $t_i = x_i m x_i^{-1}$. Each t_i fixes $x_i S$; let c_i be the point in $x_i S$ farthest from c. Each t_i fixes both c and c_i, so applying Lemma 3.4, t_i stabilizes $\mathcal{H}_c \cap \mathcal{H}_{c_i}$, where $\mathcal{H}_c$ and $\mathcal{H}_{c_i}$ are any two horospheres at c and c_i, respectively.

Pick a point p in X. Let γ be the geodesic ray from p to c, and let α_i be the geodesic ray from p to c_i. The sequence $\{t_i p\}_{i=1}^{\infty}$ is infinite, so by discreteness it must approach infinity. For all integers i, $p \in \mathcal{H}_{c,\gamma} \cap \mathcal{H}_{c_i,\alpha_i}$, so $t_i p \in \mathcal{H}_{c,\gamma} \cap \mathcal{H}_{c_i,\alpha_i}$. Therefore, in the limit as i goes to infinity, the intersection $\mathcal{H}_{c,\gamma} \cap \mathcal{H}_{c_i,\alpha_i}$ must also approach infinity.

Now let us examine the sequence $\{c_i\}_{i=1}^{\infty}$. Since $\partial_\infty X$ is sequentially compact, there is a convergent subsequence $\{c_{i_j}\}_{j=1}^{\infty}$.

Case 1: There exists a convergent subsequence $\{c_{i_j}\}_{j=1}^{\infty}$ that converges to a point $a \neq c$. Let β be the geodesic ray from p to a. The sequence of intersections $\mathcal{H}_{c,\gamma} \cap \mathcal{H}_{c_{i_j},\alpha_{i_j}}$ must then approach c, so c must be in the boundary at infinity of $\lim_{j\to\infty} \mathcal{H}_{c_{i_j},\alpha_{i_j}}$ which equals $\mathcal{H}_{a,\beta}$. Therefore, $c \in \partial_\infty H_{a,\beta}$. By Proposition 3.7, $a \in \partial_\infty \mathcal{H}_{c,\gamma}$. Since $a \neq c$, this contradicts our assumption that $\partial_\infty \mathcal{H}_c = \{c\}$.

Case 2: Every subsequence $\{c_{i_j}\}_{j=1}^{\infty}$ converges to c. Then the sequence itself must converge to c.

$$\lim_{i\to\infty} c_i = c$$

Recall that c_i is the farthest fixed point from c for a given t_i. Therefore, $\lim_{i\to\infty} x_i S = c$. Since S contains at least two points, it has non-zero diameter. Hence, the elements x_i must be acting as contractions on $\partial_\infty X$ near c, with the fixed point of these contractions limiting to c. The only isometries that behave in this way on the boundary are translations towards points approaching c. Therefore, we can apply Lemma 3.8 to derive our contradiction.

$$lim_{i\to\infty} | < t_i m^{-1} > | = \infty$$

This implies that either there exists an infinite order element, in which case T is not torsion, or there is no bound on the order of elements in T, which violates Lemma 3.3.

This completes the proof. $\square$

4 Conclusion and Future Research

We have yet to prove the general theorem that torsion subgroups of CAT(0) groups are finite. However, we have shown that in a particular case, the question can be reduced to a major open question regarding torsion groups. Many believe that infinite torsion groups with a finite number of conjugacy classes do not exist, and if this turns out to be true, then our case would be proved.

The one problem with our proof is that we have been unable to rigorously prove Lemma 3.9, which is essential for our proof. We will keep working on this part of the paper, and if a proof continues to elude us then we will try other approaches for the subcases requiring this lemma. Once this section is complete, we have a few ideas which may be applied to proving the theorem in general. In the case we proved, we assumed that $\mathcal{H}_c$ has only one point in its boundary at infinity, and we often derived our contradiction by showing that there was a geodesic in the horosphere $\mathcal{H}_c$. Even without our assumption, this information may be useful. A geodesic line in the horosphere $\mathcal{H}_c$ may imply that the CAT(0) space X is equal to the product $\mathbb{R} \times Y$, where Y is a CAT(0) space. If the CAT(0) group G acting on X has an infinite torsion subgroup, then perhaps we can construct a CAT(0) group G' which acts on Y geometrically and contains an infinite torsion subgroup T'. By induction, this would imply that X is infinite dimensional, so it cannot be proper. These are the potential directions in which we may take this research project in the coming months.

5 Acknowledgments

We are deeply grateful to our advisor, Chris Connell, who contaminated us with his endless enthusiasm and passion for the project. He was our indefatigable tour guide through the universe of negatively curved spaces, and we cannot thank him enough for all that we learnt with him this summer. We would also like to thank Kevin Pilgrim, for his excellent direction of the program, the NSF for providing this invaluable opportunity for undergraduates, and our good friend Linh Truong who provided thoughtful insights at many stages along the way.

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