Teaching Practices for Differentiating Mathematics Instruction for Middle School Students

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*The research reported in this manuscript was supported by the National Science Foundation (grant no. DRL-1252575). The findings and statements in the paper do not represent the views of the National Science Foundation.

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Abstract

Three iterative, 18-episode design experiments were conducted after school with groups of 6-9 middle school students to understand how to differentiate mathematics instruction. Prior research on differentiating instruction (DI) and hypothetical learning trajectories guided the instruction. As the experiments proceeded, this definition of DI emerged: proactively tailoring instruction to students' mathematical thinking while developing a cohesive classroom community. Analysis of 10 episodes across experiments yielded five teaching practices that facilitated DI: using research-based knowledge of students' mathematical thinking, providing purposeful choices and different pathways, inquiring responsively during group work, attending to small group functioning, and conducting whole class discussions across different thinkers. The latter three practices, at times, impeded DI. This study is a case of using second-order models of students' mathematical thinking to differentiate instruction, and it reveals that inquiring into research-based knowledge and inquiring responsively into students' thinking are at the heart of differentiating mathematics instruction.

Today's middle school mathematics classrooms are marked by increasing diversity (National Center of Educational Statistics, 2018; U.S. Census Bureau, 2015). Traditional responses to diversity are tracked classes that contribute to opportunity gaps (Flores, 2007) and can result in achievement gaps. Differentiating instruction (DI) is a pedagogical approach to manage classroom diversity in which teachers proactively plan to adapt curricula, teaching methods, and products of learning to address individual students' needs in an effort to maximize learning for all (Heacox, 2002; Tomlinson, 2005). Thus, DI involves systematic forethought rather than only reactive adaptation.

Principles of DI include that it is rooted in on-going formative assessment, positions teachers and students as learners together, emphasizes engaging all students in critical and creative thinking, and requires teachers to clarify big ideas for instruction (Heacox, 2002; Santamaria, 2009; Tomlinson, 2005). These principles are consistent with reform efforts in mathematics education (National Council of Teachers of Mathematics [NCTM], 2000, 2014), and overall differentiation is one possible, partial response to NCTM's (2000) principle of equity.¹ However, as of yet differentiating mathematics instruction has not been systematically investigated, especially at the secondary level. Indeed, secondary mathematics classrooms are places where DI is least likely to occur (Gamoran & Weinstein, 1998), in part because DI is quite demanding (Tomlinson, et al., 2003). Therefore, research is needed to learn more about how to differentiate mathematics instruction for secondary students.

In this paper we use the term *teaching practices* to refer to ways of acting that occur regularly in professional teaching and are likely to positively influence students' learning (Ball & Forzani, 2009). Following Jacobs and Spangler (2017), we note that the grain-size of teaching

¹ DI is a partial response because it does not address issues of race, culture, and socio-economic status that are involved in equity (Santamaria, 2009).

practices can vary from large (e.g., leading discussions) to small (e.g., revoicing a student's comments). In this study we aimed to understand the teaching practices used in differentiating mathematics instruction for middle school students, with these two questions:

- (1) What teaching practices are used in differentiating mathematics instruction for middle school students' diverse ways of thinking (explained in the Theoretical Frame), and how are they used?
- (2) What local theory of differentiating mathematics instruction emerges from this analysis?

This study comes from a 5-year project whose purpose was to study DI and relationships between students' rational number knowledge and algebraic reasoning. In the first two years we conducted three 18-episode design experiments, each with 6-9 seventh- and eighth-grade students. This paper comes from in-depth analysis of 10 episodes—four in Experiment 1, two in Experiment 2, and four in Experiment 3—in which students represented multiplicative relationships between quantitative unknowns. We chose to closely analyze these 10 episodes because they focused on the same topic across all experiments.

Rationale for and Research on DI

In this section we elaborate our rationale for studying DI and review relevant literature.

Rationale for DI

Tracking practices have been shown to have negative effects on students (Oakes, 1995). Low-achieving students are often placed in mathematics classes with undue emphasis on rote learning and computation, little engagement with higher demand tasks, and few possibilities for movement to classes with higher achieving peers (e.g., Boaler, Wilam, & Brown, 2000; National Research Council [NRC], 2001; Stiff, Johnson, & Akos, 2011). Furthermore, recommendations for class placements have been found to be biased against promoting students of color to advanced classes (Flores, 2007; Rubin, 2006; Stiff, et al., 2011). Students in less

advanced classes tend to fall further and further behind their higher achieving peers, a phenomenon referred to as a significant opportunity gap (Flores, 2007; NCTM, 2012).

In contrast, mathematics instruction in heterogeneous groups can help low achieving students learn more (Boaler & Staples, 2008; Burris, Wiley, Welner, & Murphy, 2008; Linchevski & Kutscher, 1998). In some studies researchers have found that high achieving students' achievement suffers in heterogeneous mathematics classrooms (Loveless, 2009; Nomi & Allensworth, 2013). However, in other studies, high achieving middle and high school students benefitted from detracked mathematics classes, where benefits included higher achievement compared to students in tracked classrooms (Boaler & Staples, 2008; Mevarech & Kramarski, 1997), increased enjoyment of mathematics (Boaler & Staples, 2008), and significant increases in earning advanced diplomas (Burris, et al., 2008). In response to such research, NCTM (2018) recently recommended that high schools discontinue tracking in mathematics.

Yet simply placing differently achieving students in the same classroom does not address the significant issues that students may experience due to their different learning needs (Gamoran & Hannigan, 2000; NRC, 2001). Teaching diverse students in one classroom must be done with great skill and care to fulfill a promise of being more humane than tracking (Mevarech & Kramarski, 1997; Rubin, 2008), and to have the potential to close opportunity gaps.

What is DI?

In our study, we had three starting points to investigate DI. First, we began with Tomlinson's (2005; Tomlinson et al., 2003) framework, which features differentiating three facets of instruction—content, process, and products—based on three characteristics of students: readiness, interests, and learning styles or preferences. In our study we aimed to differentiate for

diversity in mathematical thinking (explained in the Theoretical Framework); a definition of DI emerged from our study, and we present it in the Findings section.

A second starting point to investigate DI was ideas about responsiveness in teaching. Responsive teachers view students' ideas as central in instruction, make connections between students' ideas and disciplinary ways of thinking, and pursue students' ideas during instruction (Richards & Robertson, 2016). Researchers have examined the moves responsive teachers make (e.g., Jacobs & Empson, 2016) and how teachers learn to become responsive (e.g., Steinberg, Empson, & Carpenter, 2004). For example, third-grade teacher Kathy Staatz began her career curious about students' thinking and well-versed in Cognitively Guided Instruction (CGI) (Steinberg et al., 2004). However, in her first two years of teaching she did not inquire into students' thinking or use it in instruction. In her third year she went through four phases of change that resulted in her using students' thinking as a basis for instruction, thereby demonstrating the highest level of CGI (Franke, et al., 2001). Responsive teaching is related to DI because viewing students' ideas as central and pursuing them allows a teacher to adjust instruction to differences in those ideas.

A third starting point was specific instructional strategies for DI. In mathematics classrooms these include posing a problem in which students have a choice of numbers (Land, 2017) or posing a choice of two or three problems that address the same topic but are at different levels of sophistication (*parallel tasks*, Small & Lin, 2010). The rationale for providing choices is that students may choose work that is a good challenge for them. Students might not always make the choice the teacher would make for them, but the choices themselves are formative assessment. Land (2017) studied how 20 elementary school teachers with one or more years of CGI training used number choices to meet instructional goals. The teachers ranged from no use

of number choices to use of nuanced progressions to differentiate instruction. These latter teachers demonstrated the highest levels of instruction in the CGI framework (Franke et al., 2001). Thus, providing number choices was a tool for DI with elementary school students, supplying support for our use of choices with secondary students. Teachers may follow up the use of choices with *tiered instruction*, where the teacher designs different problems or activities for different groups of students in the class to support their learning needs (Tomlinson, 2005).

Research on and Related to DI

Since DI is relatively new, evidence of its effectiveness currently rests on many testimonials (e.g., Laud, 2011; Wormell, 2006) and some positive findings. We have found two studies in mathematics education directly about DI. First, Saxe and colleagues (Saxe, Diakow, & Gearhart, 2013) found that fourth-grade students who experienced differentiated instruction on using number lines to learn fractions and integers showed significantly greater learning gains than students in comparison classrooms. Second, Chamberlin and Powers (2010) found that prospective teachers in differentiated sections of a mathematics class for teachers showed statistically significant growth from pretest to posttest compared to those in comparison classes. Both studies are proofs of concept, but they involved populations other than secondary students.

Outside of mathematics classrooms, DI has had positive effects on elementary and middle school students' achievement (e.g., Mastropieri et al., 2006; Reis et al., 2011; Santamaria, 2009), yielded more enjoyable learning environments (Reis et al., 2011), and influenced elementary school students to demonstrate increased respect for students with disabilities and more positive cross-cultural student interactions (Santamaria, 2009). This latter finding came about in part because of the norms that need to be established when differentiating, such as respect for differences among learners and individual autonomy (Laud, 2011). Laud recommends

that teachers help students honor differences and see that *not* recognizing them in daily classroom activity would be unfair. Individual autonomy allows students to focus on their own activity amidst multiple activities in a classroom where DI is practiced. In mathematics education research, establishing the norm of deciding what counts as a different mathematical solution has been found to promote a related idea, intellectual autonomy (Yackel & Cobb, 1996), which is marked by students making decisions about correctness based on reasoning rather than the authority of a teacher or textbook. Therefore, working on norms specific to mathematics classrooms could support the norms advocated for in DI.

We address two more, related recommendations about DI. One is that teachers group students in different arrangements for different purposes—for example, sometimes students who think similarly should work together, and sometimes groups should be more heterogeneous (Tomlinson, 2005). If students are working in groups, they are likely to need support; without training on how to work together in groups, secondary students often curtail opportunities for mathematics learning and participation (e.g., Barron, 2003; DeJarnette & Gonzalez, 2015; Horn, 2012). Cohen and colleagues (Cohen, 1994; Cohen & Lotan, 1995) have developed a method to address this issue, termed *complex instruction*, where teachers emphasize that everyone has multiple, varied abilities that can contribute to solving mathematical problems; assign competence to low status students; and support students to interact as they work on groupworthy tasks. Support includes assigning roles to students in small groups (Featherstone, et al., 2011).

Second, in DI the goal is not to group students into permanent tracks within a classroom but for students to have frequent contact across differences (Tomlinson, 2005). Class discussions are one vehicle for such contact. Leading class discussions based on students' mathematical thinking requires navigating differences in that thinking (e.g., Heaton, 2000; Lampert, 2001;

Staples, 2007). Indeed, Staples found that a ninth-grade teacher of an inquiry-oriented pre-algebra class “pursued discrepancies” (p. 181) in students’ thinking in ways that teachers following traditional patterns of instruction did not. Staples identified this teaching action to be part of a broader component, “establishing and monitoring a common ground” (p. 180), one of three components of the teacher’s instruction. The emphasis of this component seems to be on maintaining commonality: Differences were pursued so that they could be eliminated in favor of “common ground.” An open question for us was in what ways students with diverse ways of thinking who had worked on different, related problems could find common ground.

Theoretical Framework

In this section we discuss how we framed the connection between students’ thinking and classroom interaction, diversity in students’ thinking, and the broad mathematical topic of our three experiments. To do so, we present our approach to mathematical thinking and interaction, students’ multiplicative concepts, and algebraic reasoning.

Mathematical Thinking and Interaction

We view students’ mathematical thinking as *first-order knowledge* (Steffe, 2010), or the ways and means of operating that a student has available to engage in mathematical activity. Since we do not have direct access to first-order knowledge, we organize our experiences with students’ mathematics by describing and accounting for it using our constructs: operations, schemes, and concepts. Operations are the components of schemes, goal-directed ways of operating that consist of a situation as conceived of by a learner, activity, and a result that the learner assesses in relation to her goals (Steffe, 2010; von Glasersfeld, 1995). For us, mathematical learning means a learner is making reorganizations, or accommodations, in her schemes in on-going interaction in her experiential world. When students interiorize, or re-

process of the results of their schemes so that they are available prior to activity, they abstract concepts (von Glasersfeld, 1982).

We refer to webs of these constructs as *second-order knowledge, or models* (Steffe, 2010; Steffe, von Glasersfeld, Richards, & Cobb, 1983). We note that we cannot say that students necessarily “have” the operation of iterating, a mental action of repeatedly instantiating a unit toward a goal. However, if we interact with students who regularly repeat a unit fraction to make a larger fraction, we model this activity with the operation iterating. Following Steffe (2010), we regard second-order models as co-constructed with students and as instruments of interaction.

We view interaction in terms of two non-intersecting domains (Steffe, 1996): intra-individual interactions of constructs within a person, such as accommodations in schemes, and individual-environment² interactions of which social interactions are a major part. Social interaction can open possibilities for accommodations and make schemes and concepts apparent via verbalizations, non-verbal interaction, drawn representations, or mathematical notation. Similarly, the construction of particular schemes and concepts can dramatically influence how a student interacts with others in a classroom (e.g., Hackenberg, Jones, Eker, & Creager, 2017). However, interaction of a particular kind in one domain does not directly cause interaction of a particular kind in the other (Davis & Sumara, 2006; Steffe, 1996).

Students’ Multiplicative Concepts

Broadly speaking, students enter middle school operating with three different multiplicative concepts that significantly influence students’ rational number knowledge (Hackenberg, 2010b; Norton & Wilkins, 2012; Steffe & Olive, 2010) and algebraic reasoning (Hackenberg & Lee, 2015; Olive & Caglayan, 2008; Tillema, 2014). We used these concepts as

² The environment of a person is what an observer sees that does not include the person (Steffe & Thompson, 2000).

markers of diversity in thinking. Students' multiplicative concepts are the interiorized results of students' units coordinating schemes (Hackenberg & Tillema, 2009; Steffe, 1994). Here a *unit* means a unit of measurement, including non-standard units (Ulrich, 2015); conceiving of a quantity as a *unit of units* means thinking of a length as, for example, both 1 foot and 12 inches.

Students operating with the first multiplicative concept (MC1 students) can view a length as unit of units, but they have to actually make parts rather than imagine them. In addition, they do not experience a logical necessity to reunite small units into a larger containing unit (Hackenberg, 2013; Steffe & Olive, 2010). For example, MC1 students can share a 1-meter length equally into five parts by making marks, but after these students cut a length into parts, they no longer think of the parts as parts of the whole length—they do not maintain that part-to-whole relationship.

Students operating with the second multiplicative concept (MC2 students) can view quantities as units of units, and they can make three levels of units as they act. In contrast with MC1 students, MC2 students can imagine a 1-meter length partitioned into five equal parts without physically making the partitions, so they can think of the length as a unit of five units (Steffe & Olive, 2010). In the process of solving a problem they might partition each of the five parts into eight mini-parts to produce a 1-meter length containing 40 mini-parts. They have created three levels of units in their activity: The 1-meter length is a unit of five units, each containing eight units. However, in further activity, these three levels of units are not salient for the students, and they work with two levels of units. Thus, in further activity they conceive of the 1-meter length as (only) a unit of 40 units.

In contrast, MC3 students can take three levels of units as given and flexibly switch between three-levels-of-units structures (Steffe & Olive, 2010). In the 1-meter example, MC3

students can operate as MC2 students do, but they can maintain a view of the length as a unit of five units each containing eight units, and they can switch to viewing the length as a unit of eight units each containing five units. Transitioning between these three concepts requires a substantial reorganization of thinking, and students may remain at a stage for two or more years (Steffe & Cobb, 1988; Steffe & Olive, 2010).

A Quantitative Approach to Algebraic Reasoning

Our study focused on students' *algebraic reasoning*, which we take to be about (1) abstracting and generalizing arithmetical and quantitative relationships, and systematically representing those abstractions and generalizations; and (2) learning to reason with algebraic notation in lieu of quantities (Kaput, 2008; Stephens, Ellis, Blanton, & Brizuela, 2017). In our experiments we took a quantitative approach to algebraic reasoning because of its fruitfulness for supporting algebraic reasoning (Ellis, 2007; Hackenberg & Lee, 2015; Tillema & Gatza, 2017).

We follow Thompson (1993, 2011) in conceiving of a *quantity* as a property of one's concept of an object or phenomenon, along with a measurement unit and process. For example, to conceive of a person's height requires conceiving of a unit and of the distance subdivided into these units; counting is one process to enumerate these units to find a value. We conceptualize a *quantitative unknown* as the potential result of measuring a fixed but unknown quantity before actually measuring it (Steffe, Liss, & Lee, 2014). In our experiments we focused primarily on distances. Thus, an unknown could be thought of as a distance for which we have a measurement unit; we can imagine subdividing the distance into those units; but we do not know how many of those units will be needed (Figure 1).

Method

Participants

Steffe (2017) estimates that at the start of sixth grade, 30% of students are operating with each of the first and second multiplicative concepts, and 40% of students are operating with the third. Thus, we aimed to have three MC1, three MC2, and three MC3 students in each experiment. We identified MC1 students in our selection process, but they did not opt to participate. Therefore, we invited nine students to participate in each experiment: six MC2 students and three MC3 students. Due to attrition, 22 students completed the study: Nine, seven, and six students across Experiments 1, 2, and 3 respectively. We now describe our preparation for the experiments, the selection process, the measures, data collection, and data analysis.

Preparation

To prepare for the first experiment, we articulated six goals for students' mathematical thinking informed by prior research, with conjectures and essential questions (Heacox, 2002). For example, the first goal was: Students will construct a quantitative unknown. We conjectured that this construction was possible for both MC2 and MC3 students. The essential question was: What is a quantitative unknown and how can it be represented?

We also created hypothetical learning trajectories (HLTs) (Simon, 1995) for students operating with each multiplicative concept (Table 1). An HLT consists of a second-order model of students' current ways of thinking, a learning goal, and an account of how the learning may proceed, including sequences of problems and supportive questions (cf. Simon, 1995; Steffe, 2004). The HLTs across different thinkers were connected. For example, for the MC3 students we created an HLT for reciprocal reasoning (Appendix A), and for the MC2 students we hypothesized that reciprocal reasoning with unit fractions (but not all fractions) was possible (Appendix B). Thus, we hypothesized that MC2 and MC3 students could work on different but related problems when aiming to learn to reason reciprocally, and that we could have class

discussions about the problems. The goals, conjectures, essential questions, HLTs, and other related planning documents guided both lesson planning and spontaneous decision-making during episodes. The HLTs for reciprocal reasoning guided the instruction for the 10 episodes that we analyzed for this paper.

We also articulated conjectures about DI and wrote five goals for DI. At the start of Experiment 3, these goals were: Students will have choices in their work; students will experience instruction tailored to their ways of thinking; students will support everyone's learning while building a community; students will learn to communicate about big ideas even when they have worked on different problems; students will experience structured exposure to different ways of thinking and consider how that can contribute to their own learning.

Selection Process

To launch each experiment, we observed in classrooms at a local middle school and then implemented a two-part selection process with 21-23 seventh- and eighth-grade students: a 30-minute interview (Appendix C) and a 12-item worksheet (Appendix D). We observed in classrooms because we wanted participants who would be willing to speak in our class. In addition, our presence in classrooms allowed students to begin to know and trust us.

Measures

The interview questions and worksheet were designed to assess students' multiplicative concepts using problems involving nested quantities and students' fractions knowledge using problems designed to assess fraction schemes (Norton & Wilkins, 2012). Interviews were analyzed based on prior research (Hackenberg, 2010b). The worksheet was analyzed using published guidelines (Norton, Boyce, Phillips, Anwyll, Ulrich, & Wilkins, 2015; Norton & Wilkins, 2012). We give a brief summary of how we assessed students' multiplicative concepts

based on one interview problem; our final assessments were made from students' work on multiple problems.

In the selection interview, all students worked on the Crate Problem (Appendix C, #1). Some students drew a single picture in which cans were nested within packages, packages within boxes, and boxes within the crate. These students also computed 4×8 as the number of cans in a box, 32×6 as the number of cans in a crate, and they had a meaning for 6×8 as the number of packages in the crate. So, they flexibly switched between units containing other units as they worked on the problem. We assessed these students to be operating with the third multiplicative concept. In contrast, some students did not draw a single picture of all embedded units—for example, they drew a box consisting of eight packages of four cans each, and then a separate picture of a crate containing six rectangles. These students often conflated containing units in their work on the problem. For example, these students might see the rectangles in their drawing as packages at one point and then later refer to them as boxes, indicating that only six of them should be included in a crate. We assessed these students to be operating with the second multiplicative concept. Still other students did not draw all nested units and did not arrive at an answer to the problem. We assessed these students to be operating with the first multiplicative concept. Students' work on the written assessment (Appendix D, #7) was consistent with their work in the selection interview.

Data Collection

The 18 episodes of each experiment ran after school on Tuesdays and Thursdays. Each episode lasted 1 hour and was video recorded with one stationary and two roaming cameras. The first author served as the teacher for the episodes. Students worked on problems and activities in groups of two or three, often using a software program called JavaBars (Biddlecomb & Olive,

2000). Screenflow software (Telestream LLC, 2013) captured students' computer work. During episodes the teacher interacted with students in groups and held class discussions about mathematical ideas based on goals, essential questions, and students' work. Other team members operated roaming cameras, observed, took notes, and interacted with students in their groups.

Between episodes the team processed video data; wrote a summary, observations, and conjectures for each episode; and watched video and took notes. The teacher also kept a reflective research journal. At weekly meetings the team discussed observations and conjectures in order to decide on student groups, problems, and discussion foci for coming episodes. Some problems were designed before the experiments, but all evolved as the experiments proceeded. Following each experiment each student participated in a 45-minute, video recorded interview to assess the student's understanding of topics from the experiment and experience of the class.

Selection of Data for Analysis

To analyze how DI functioned in the experiments, we thought it best to examine the same topic in each experiment. Only two topics were consistent across all three experiments: quantitative unknowns at the start of each experiment, and multiplicative relationships between two quantitative unknowns, which we refer to as Two Unknowns Problems. We worked on these problems from episodes 9-12 in Experiment 1, 10 and 12 in Experiment 2, and 11-14 in Experiment 3. For this paper we chose to closely analyze these 10 episodes because they better represented the implementation of DI than beginning episodes. At the beginning we had introductory tasks to accomplish, such as students taking pre-assessments and learning to use technology, as compared to later episodes when students were into the flow of the class.

Data Analysis

For this paper we coded our data, using some a priori codes and mostly emergent codes. For example, we went into the analysis knowing that we were looking for moments when the teacher appeared to be posing problems that were in harmony with one or more students' ways of thinking, as well as when the teacher was posing problems or questions that presented a challenge, aiming to assess if that challenge was overwhelming or bearable (Hackenberg, 2010a; Tzur, 1995). Yet most of the codes emerged in the spirit of grounded theory (Corbin & Strauss, 2008). This analysis was appropriate because we had some prior research and conjectures on which we were building. However, we did not have a framework or theory of DI in mathematics education to follow—we were aiming to create a humble theory (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) from our data.

We engaged in three rounds of coding: initial coding, process coding, and then focused coding to elicit themes (Saldana, 2013). We began with an initial, open coding of transcripts from episodes 9-11 from Experiment 1, watching video in conjunction with transcripts to make sounder inferences. This initial coding occurred between Experiments 2 and 3; its purpose was to characterize DI in Experiment 1. We met regularly to discuss and revise our codes and then collapsed them thematically, yielding 24 codes.

Most of these codes were processes because we were trying to understand the actions and interactions of the teacher and students. A process “is ongoing action/interaction/emotion taken in response to situations, or problems, often with the purpose of reaching a goal or handling a problem” (Corbin & Strauss, 2008, pp. 96-97). This choice was appropriate because in our theoretical frame, individual-environment interactions are critical for learning. Therefore, some of our process codes marked individual-environment interactions, such as “students experiencing difficulty communicating in a group” and “teacher or students identifying ways of thinking as

similar or different.” In addition, we were trying to uncover teaching practices, which are made from actions that teachers regularly take, guided by their reasoning and knowledge (Ball & Forzani, 2009). Thus, some process codes marked actions of the teacher, such as “teacher eliciting multiple student voices or thoughts.”

Following Experiment 3, we used these 24 codes in ATLAS.ti to perform a second, process coding on the 10 targeted episodes. We also coded planning and reflective documents for the episodes, including goals, HLTs, lesson plans, conjectures, and sections of the teacher’s research journal. Because we were capturing teacher actions, we still used primarily process codes, such as “teacher using purposeful grouping” and “teaching planning for different ways of thinking.” At least two researchers coded every document, and we met regularly to discuss and revise our codes, expanding the number of codes to 38. We wrote data summaries, memos (Corbin & Strauss, 2008) consisting of interpretations of data, and conjectures about the development of DI in the 10 episodes.

Finally, we used focused coding (Saldana, 2013) to generate five themes from our codes. Focused coding is a later analytical coding process in which codes are grouped into themes, and then the codes and themes are reviewed for coherence in relation to the entire body of data. For example, 14 codes marked the teacher’s actions when working with students in groups; we organized these into a theme. We discussed the themes and how codes mapped to themes, and then we made revisions. To check the themes and codes across the data, we examined the codes across the three experiments to check for trends. All codes were used regularly in all three experiments, and we concluded that the themes and codes captured the data well. Following these procedures, we organized the themes into a local theory of differentiating mathematics instruction for middle school students, presented in the final section of the paper.

Findings

Our analysis revealed five practices that facilitated DI (Figure 2). Practices 3, 4, and 5 also, at times, impeded DI. With these practices, a definition of DI emerged: *Proactively tailoring instruction to students' mathematical thinking while developing a cohesive classroom community*. Tailoring instruction requires posing problems that are in harmony with students' thinking, posing challenges at the edge of students' thinking, as well as questioning in a responsive way. A cohesive classroom community is students and a teacher who work together to foster everyone's mathematical learning, and whose diversity is excavated and valued. We now describe each practice, giving examples from the data that illuminate it and explaining how the practice facilitated or impeded tailoring instruction and building classroom community. We present the practices roughly in order of how they might appear in the unfolding of a lesson.

Practice 1: Using Research-based Knowledge of Students' Mathematical Thinking

Initially the teacher conjectured that all MC3 students would construct reciprocal reasoning with quantitative unknowns by making accommodations in their iterative fraction schemes. Students who construct iterative fraction schemes conceive of fractions as multiples of unit fractions. For example, for these students $7/5$ is $7 \times 1/5$, as well as 1 and $2/5$. Prior research has found that conceiving of fractions in this way requires operating with the third multiplicative concept (Steffe & Olive, 2010). In contrast, the teacher conjectured that MC2 students would construct reciprocal reasoning with only unit fractions because these students have yet to construct iterative fraction schemes (Steffe & Olive, 2010). Although retrospective analysis revealed that these conjectures needed some revision, at the time of the experiments the HLTs were a research-based guide for lesson planning and interaction during episodes, helping the teacher tailor instruction to students' thinking.

For example, the teacher conjectured that both MC2 and MC3 students could work on representing two unknown heights where one height was a whole number multiple of the other, such as the **Corn Stalk Tomato Plant Heights Problem (CS TP Hts Problem)** (Figure 3, top). However, in a prior study some MC2 students did not draw pictures that showed precise multiplicative relationships (from our perspective), and most did not write multiplicative equations (Hackenberg & Lee, 2015). Therefore, in the current study the research team designed problems and questions to support MC2 students to eliminate these difficulties. Since prior research indicates that MC2 students can conceive of a length as a unit to iterate (Steffe & Olive, 2010), we conjectured that questions about how to use the smaller height to make the larger height would support MC2 students to revise their pictures by iterating the small height five times to draw the larger height. In addition, if students did not write multiplicative equations, the teacher had a series of planned questions about students' notation, frequently asking students how their equations made sense in their pictures (Appendix B).

We conjectured that MC3 students were poised to represent in pictures and equations two unknown heights with a fractional relationship between them, such as the **Fern Sunflower Heights Problem (FSH Problem)**, Figure 3, bottom), and to construct reciprocal reasoning. However, we did not know how common or durable this construction would be. Therefore, we designed problems and questions to support MC3 students to construct reciprocal reasoning. For example, if in the **FSH Problem** students did not write an equation with $5/3$, we planned to ask them how to determine the fraction the larger height was of the smaller height (Appendix A). As the experiments proceeded, we revised questioning support in the HLTs for all students.

Using and revising the HLTs, one form of research-based knowledge of students' thinking, facilitated DI in these experiments because it guided our design of problems and

questions, allowing us to tailor instruction to students' ways of thinking. Also, because the HLTs for MC2 and MC3 students were related, they were a basis for conducting whole class discussions across different thinkers (practice 5), discussed later. More broadly, we suggest that using research-based knowledge of students' mathematical thinking facilitates DI.

Practice 2: Providing Purposeful Choices and Different Pathways

Across the experiments, the teacher used *parallel tasks* (Small & Lin, 2010) to introduce work on Two Unknowns Problems (Figure 3). Students were given both tasks and asked to choose one that would be a good level of challenge for them. Students' choices and work on these problems informed how we organized small groups in subsequent episodes. We use Experiment 1 as an example.

In episode 9 of Experiment 1, all students chose the **CS TP Hts Problem** but worked on the problem in quite different ways. We observed three types of responses (Figure 4). Three MC2 students did not, from our perspective, show a 5 times relationship between the two heights in pictures or equations (group I); two MC2 students drew fairly accurate pictures but left us with some questions (group II).³ For example, one student drew a picture that looked reasonable (Figure 4, upper middle). However, when asked to show the 5 times relationship, he partitioned the longer height into six parts, not five (Figure 4, lower middle). Finally, three MC3 students showed a 5 times relationship accurately from our perspective (group III).

In the next episode we put students into groups I, II, and III, and they worked in these groups for the next three episodes. Based on our HLTs and our understanding of student thinking to this point, we tiered instruction. We designed questioning support for groups I and II based on

³ One MC2 student was absent in episode 9.

our HLTs, posed the **FSH Problem** to group III, and designed a different sequence of follow-up problems for each group.

For group I, we questioned students about how they could use the small height to check their drawings for the **CS TP Hts Problem**. This questioning was effective in that the students in group I successfully engaged in a revision of their pictures by iterating the small height. Then we posed a problem with a 3 times relationship between the heights of a father and his 1-year-old son, using a smaller whole number relationship, followed by a problem with a $\frac{1}{4}$ relationship between the heights of a puppy and its mother. In both problems imagining the objects themselves could allow students to determine which height was smaller.

For group II, we planned similar questioning support on the **CS TP Hts Problem** but conjectured that students would make quicker progress than students in group I. We were not entirely correct, in that students in both groups demonstrated similar constraints. Then, for these students we posed a problem with a $\frac{1}{4}$ relationship between two plant heights, followed by a $\frac{2}{5}$ relationship between the heights of two dogs. In these problems, imagining the objects did not allow students to determine which height was smaller.

For group III, we planned questioning support for the **FSH Problem** based on our HLT (Appendix A). This questioning was effective in that two of the three students made the initial construction of reciprocal reasoning in episode 10. Then we planned follow-up problems with other fractional relationships between the two heights: $\frac{2}{7}$, $\frac{7}{5}$, and $\frac{13}{27}$. These follow-up problems were intended to support the students to stabilize their reciprocal reasoning, and they also evolved across the experiments based on our on-going analysis of student thinking.

Thus, the combination of providing purposeful choices to students through parallel tasks, interpreting students' work on the problems, and designing pathways via tiering instruction

allowed us to tailor instruction to students' learning needs, facilitating DI. The contribution of this practice to building a community is more tenuous. It has the potential to do so because it can build trust that the problems the teacher poses are helpful for different thinkers, an important component of building community. However, it also can surface differences that can be hard to navigate, as we demonstrate when we discuss practice 5.

Practice 3: Inquiring Responsively During Group Work

This practice occurred in every episode, and it could support or impede DI depending on how well the teacher enacted it. We will look at the Two Unknowns work in Experiment 3, which began with parallel tasks (Figure 3). We focus on the pair of MC2 students, Heather₂⁴ and Milo₂, who chose the **CS TP Hts Problem**. To preview the story: Inquiring responsively based on conjectures worked well for Milo₂, while for Heather₂ the teacher was initially blinded by conjectures and had to re-focus on seeing Heather₂'s ways of operating anew, with fewer assumptions. For this reason, the section focused on Milo₂ is significantly shorter than the section focused on Heather₂.

Milo₂. Based on experience over two experiments, the team had developed three conjectures for how MC2 students would work on the **CS TP Hts Problem**:

- 1) MC2 students will not represent the 5 times relationship precisely from our perspective but will instead draw relationships such as 2 or 3 times.
- 2) (Alternative to 1) MC2 students will use known values to make their drawing, thereby not representing unknowns.
- 3) MC2 students have constructed a length as a unit they can iterate, which they can use to revise their drawings to show a precise multiplicative relationship from our perspective.

⁴ All names are pseudonyms; we use subscripts to denote students' multiplicative concepts at the time of the study.

For Milo₂, the first and third conjectures were borne out in that he began by drawing a picture on paper and then made a JavaBars picture that showed, from our perspective, a 3 times relationship (Figure 5). The teacher questioned Milo₂ about whether the picture showed the 5 times relationship from the problem. He said yes. When asked how it showed that, Milo₂ responded by iterating the smaller height on his paper drawing (Figure 6, left) and said, “I should have made the cornstalk a little bit higher.” The teacher asked if he could show that, and he made a new drawing that demonstrated the corn stalk height spanning five iterations of the tomato plant height; later he made a JavaBars picture (Figure 6, right). Thus, in this interaction the teacher inquired responsively with Milo₂. Indeed, based on engaging in practices 1, 2, and 3 for two prior experiments, she did so efficiently and thus tailored instruction to Milo₂.

Heather₂. In contrast, Heather₂’s drawing was different from Milo₂’s and not typical for MC2 students in the experiments to this point. Heather₂ drew a tomato plant height that was about half the corn stalk height from our perspective; then she made five evenly spaced marks on the upper part of the corn stalk height (Figure 7). Upon viewing the picture, the teacher conjectured that Heather₂ was adding five more equal parts onto the tomato plant’s height because sometimes students interpret a 5 times relationship as requiring five more parts than the given part, or six parts total (Olive & Steffe, 2010).

When asked about her work, Heather₂ said, “I have five lines.” Based on the teacher’s conjecture, the teacher did not use Heather’s language of lines and instead questioned Heather about the five *parts* above the lowest part on the corn stalk bar. Heather₂ said they showed “five times as higher.” Then she drew a picture in JavaBars, with teacher support. When she created the JavaBars picture (Figure 8, left two bars), she said, “It’s not five, though; it’s four.” The teacher counted the five parts above the grey tomato plant height, and Heather₂ seemed to agree.

Yet when the teacher left, Heather₂ took a copy of the corn stalk height bar and made five marks above the grey part, making six small parts (Figure 8, right). However, her parts were not equal. “That sucks,” she said and erased this bar. At this point she seemed satisfied with her drawing because she saved it and went on to other parts of the problem.

The data presented so far is fairly strong evidence that Heather₂ was representing the 5 times relationship with five evenly-spaced marks in the upper part of her corn stalk height bar. First, she described her drawing as involving “five lines.” Second, when the teacher supported her to make a drawing in JavaBars, and they arrived at a picture (Figure 8, left), Heather₂ said it was not five but four. That picture does show four *marks* above the mark that defines the tomato plant height. Third, she tried putting five marks in the upper part of a copy of the corn stalk height bar (Figure 8, right) and then rejected that, apparently because the parts were not equal, and perhaps because the teacher had been emphasizing five parts. We note that Heather seemed to want her marks to be evenly spaced. However, it is not clear that she was aiming to create five parts beyond the part that ostensibly represented the tomato plant height. Therefore, it’s not clear that she was working with an image of an unknown height that was measurable in the sense she anticipated it being segmented into an equal number of units—let alone with the idea that there was a multiplicative relationship between a single unit of length and the larger length (Thompson & Saldanha, 2003). At the time the teacher did not see all of this, and trying to be responsive based on her conjecture blinded her to seeing that Heather₂ was likely counting marks, not parts. In this way the teacher did not inquire responsively enough into Heather₂’s thinking and did not tailor instruction as well for Heather₂ as she might have.

The story of Heather₂ does not end here: The teacher and her groupmates interacted about these issues for two more episodes, with Heather₂ repeatedly commenting that her groupmates’

pictures (e.g., Figure 6) did not make sense. During the next episode the teacher realized Heather₂ was thinking quite differently from prior students, and she began to investigate Heather₂'s thinking with fewer assumptions. We use Milo₂ and Heather₂ to make the following point: In some cases, following conjectures works smoothly to tailor instruction to students' thinking, particularly when there is evidence and theory behind the conjecture as in Milo₂'s case. Thus, this practice can support DI. However, conjectures followed too closely can interfere with fully seeing students' thinking, which can impede DI. Thus, the teacher has to hold conjectures lightly so that they do not become blinding. Not surprisingly, this task is difficult when balancing many groups. Yet with new learning of student thinking comes new ways of thinking to watch out for with future students. In addition, as the teacher cultivates this second-order knowledge, she can more easily build trust with a range of students about posing tasks that are a good challenge, which supports the development of community.

Practice 4: Attending to Small Group Functioning

To work on building community across different ways of thinking, the teacher began each experiment with a discussion of how students in the same class have different strengths and areas to work on. The teacher emphasized that all students have something to contribute to everyone's learning, including her own, and she returned to this issue several times during each experiment. She explained that students would work in groups at most class meetings in order to talk about ideas together, and these groups would change frequently.

Across the experiments the small groups varied in how students got along and talked about their ideas. In Experiment 1 the variation was considerable. When groups supported students' thinking and work, group members tended to listen to each other and ask questions about what others said; in some cases, they tried to use what someone else had said in their own

thinking. Group members sometimes disagreed, but they did not seem to view disagreement as a referendum on who was correct or smart. This kind of group work seemed to lead to feelings of connection between students and generally facilitated DI.

In contrast, when groups had difficulty working together, it impeded DI. For example, in Experiment 1 Gabriel₃, Martin₃, and Stephanie₃, experienced frequent difficulties. One characteristic of their interaction was that they tended to work individually and come together for brief time periods (cf. Barron, 2003). Second, their interaction was territorial: They would verbally spar over who was going to use the computer and who made what contributions. Third, the boys excluded Stephanie₃. She was not easy to include because she often focused on having her own work recognized as “smart” and the best (cf. Horn, 2012). In addition, the boys seemed to like each other, despite the sparring. Even when they did try to include Stephanie₃, she seemed to feel excluded. When the teacher came over in the middle of episode 10, Stephanie₃ remarked, “We’re having team work issues.” The teacher facilitated discussion—and in fact, near the end of the episode both Martin₃ and Gabriel₃ made the initial construction of reciprocal reasoning. However, the difficulties persisted and seemed to interfere with at least Stephanie₃’s learning and with feelings of connection between the students.

In response to these issues within Experiment 1, the teacher instituted “red cards” that groups could put out when they needed the teacher to come by (Heacox, 2002). This change was helpful. In addition, the teacher used specific patterns of moves as she circulated: At each group she questioned students to make their thinking more visible, elicited multiple student voices when all were not contributing, and suggested tasks for the group before she moved on. However, these moves did not uniformly support productive group work in Experiment 1.

Therefore, in Experiments 2 and 3 we instituted some changes to better support group work. We followed Featherstone and colleagues (Featherstone, et al., 2011) by defining roles for group interaction. Since we had groups of three, we used facilitator, questioner, and recorder/reporter. We discussed these roles in the second episode of each experiment as part of a discussion about what behaviors could support and limit group work. We instituted calling a teacher over for group questions only (Featherstone, et al., 2011), which meant that students needed to ask groupmates first, thereby supporting groups to explore ideas together. We also engaged in periodic check-ins called “Overheard Last Time,” (cf. Featherstone, et al., 2011) in which we looked together at some anonymous statements from groups during the last episode and examined whether they supported or limited the group’s work and why. In addition, the teacher, in circulating, augmented her repertoire of moves by encouraging students to enact group roles where appropriate (e.g., suggesting that the questioner ask a “why” question).

Although the groups in Experiments 2 and 3 continued to vary in how well students talked about ideas together, no group had the same level of difficulty as in Experiment 1. Evidence for this claim is the sharp decrease in codes for “students experiencing difficulty communicating in a group” and “students expressing issues about smartness and/or competence” from Experiments 1 to 2, which was maintained in Experiment 3. In addition, students seemed to understand better that we were there to listen and work through ideas, not just to get correct answers. For example, in episode 15 of Experiment 2 we had a class discussion about mixtures of lemonade powder and water that would make lemonade with the same flavor as a pitcher of 3 T powder with 16 ounces of water. Students had just discussed that $1\frac{1}{2}$ T powder and 8 ounces of water would work because it made one-half of the pitcher. When asked for another mixture, Elliot₂ proposed 2 T powder with 8 ounces of water. Brad₃ tried to phrase a question to Elliot₂,

but Danielle₂ interrupted, stating, “that’s wrong!” Steven₂ immediately spoke up in defense of Elliot₂, saying, “No, let’s hear everyone out first!” Thus, at least Brad₃ and Steven₂ showed evidence of carrying out the norms of listening to and working through each other’s ideas.

In sum, the change in how we supported group work in Experiments 2 and 3 made space for tailoring instruction to occur because it allowed students to work on their tasks while the teacher inquired responsively, practice 3. The classroom was less distracting; students were more focused and got more work done. More broadly, we propose that in Experiments 2 and 3 we identified the importance of *group autonomy* and supported students to work toward it. We define group autonomy as being able, as a small group, to work independently but together with other groups in the classroom, taking responsibility for tasks and making some judgments about solutions and processes without appeal to an authority such as the teacher (cf. Yackel & Cobb, 1996). In addition, the norms we developed for small groups contributed to building classroom community because students seemed to understand that listening to different perspectives was important, including in whole class discussions.

Practice 5: Conducting Whole Class Discussions Across Different Thinkers

This practice brings up the issue of what students have the potential to learn from each other, particularly if their ways of thinking are significantly different. For example, sometimes MC2 students cannot interpret MC3 students’ ideas (e.g., Hackenberg, 2010b), and sometimes they can. However, when they do, that does not mean that they have constructed the operations, schemes, and concepts necessary to generate those ideas themselves, and so they may construct different schemes from MC3 students (e.g., Hackenberg & Tillema, 2009). Similarly, MC3 students sometimes cannot see the reason that MC2 students would think the way they do (e.g., Hackenberg et al., 2017), although they can notice and develop an interest in differences (e.g.,

Hackenberg, 2010a). We used these observations to develop three conjectures about how students might influence each other in discussions in relatively positive ways:

- 1) Students may interpret other students' mathematical ideas and find them sensible, so they get exposed to ideas to consider. However, MC2 students may not understand or construct the ideas of MC3 students due to differences in schemes and concepts, and MC3 students may be equally baffled by why MC2 students would think the way they do.
- 2) If students find another's mathematical idea sensible or useful after they have tried to understand it, students may adopt the idea because they construct, or have constructed, the schemes and concepts necessary for it.
- 3) If students do not find another's mathematical idea sensible or useful after they have tried to understand it, they might still be enlarged by the attempt to understand other humans in the world. However, it is also possible that they will just be puzzled.

We found that similar to practices 3 and 4, practice 5 sometimes impeded DI and sometimes supported it. When it impeded DI, as in our first example, it threatened the development of a cohesive classroom community. When it supported DI, as in our second example, it allowed the teacher to tailor instruction to student thinking and develop community.

Episode 12, Experiment 1. At the beginning of episode 12, the teacher posted student work on Two Unknowns problems from episode 11. Each of two groups of MC2 students had worked with the relationship $\frac{1}{4}$, and the group of MC3 students had worked with the relationships $\frac{2}{7}$ and $\frac{7}{5}$. There were four absences, so in class were Connor₂, Lucy₂, Tim₂, Gabriel₃, and Martin₃. The teacher stated that not everyone had worked on the same problem but that all problems involved two heights, and they were going to discuss the $\frac{1}{4}$ relationship. She

pointed out one group's equation, " $q/4 = c$ " where q represented the larger height and c the smaller. The teacher asked, "can you write the equation $q/4 = c$ using multiplication?"

Connor₂ suggested " $4c = q$." The teacher acknowledged that this equation used multiplication, but she was wondering if it was possible to multiply the larger height by something to produce the smaller height. Gabriel₃ immediately raised his hand, but the teacher asked him to wait in order for all students to think. Connor₂ said no. Tim₂ said, "Unless you do two steps." He was not sure what to do and then said it was possible, but he did not know how. Connor₂ suggested "multiplying it by itself" and then dividing it. As the MC2 students spoke, both Martin₃ and Gabriel₃ put their hands up high. When the teacher held off on their input and repeated her question about whether it was possible to write $q/4 = c$ using multiplication, Connor₂ said emphatically, "No!" Lucy₂ did not suggest ideas and nodded when the teacher asked if it was a strange question. Tim₂ suggested using scientific notation, since that was what they were learning in his math class. Thus, at this point, the question appeared to be puzzling for MC2 students, while MC3 students continued to raise their hands to respond.

The teacher called on Gabriel₃, who proposed " $\frac{1}{4} \times q = c$." Martin₃ agreed. The teacher asked whether that seemed sensible, requesting that they "be honest." Lucy₂ said it did, "Because you would have to, um, multiply by the reciprocal which would be four, and then you multiply the reciprocal on other side and you get the same exact answer." Her reason seemed to focus on rules for creating equivalent equations. Connor₂ mentioned fractions, turned a little red, and said he felt bad he had not said something about fractions before. Gabriel₃ patted him on the back.

Then the students discussed what multiplying by a fraction and dividing by a fraction meant. Martin₃ said, "multiplying by $1/4$ and dividing by 4 is the same thing." The teacher asked how many agreed. Gabriel₃ did, while the others put their thumbs "in the middle" to indicate

“sort of.” A short while later the class was considering $q \div \frac{1}{4} = c$, because Lucy₂ had written this equation on her paper in the prior episode (incorrect from our perspective). The teacher asked how they saw the equation in the picture. Lucy₂ smiled and shrugged. Connor₂ said, “if you were to divide the larger height by $\frac{1}{4}$, it’d be like dividing it by 4, so then it’d equal the smaller height.” The teacher asked whether he saw dividing by $\frac{1}{4}$ and 4 as the same and Connor₂ said, “Yeah!” Martin₃ and Gabriel₃ immediately disagreed.

Thus, the whole class discussion in episode 12 surfaced quite different ways of thinking. Although the students had voiced sharply contrasting ideas before, this discussion seemed like a bigger challenge to the development of community because the viewpoints did not seem easily reconciled. The MC3 students seemed to know immediately how to turn a division equation into a multiplication equation (without complete justification, but the idea seemed sensible to them). In contrast, the MC2 students seemed quite puzzled by the question and, despite Lucy₂’s comment about creating equivalent equations, seemed to have few reasons to agree with the MC3 students’ ideas. Thus, the MC3 students appeared to have knowledge that the MC2 students did not. Furthermore, it seemed that it would not be a quick process to develop a sound conceptual foundation for these ideas. In terms of our conjectures, conjecture 3 seemed to be dominantly at play. The teacher was concerned about setting up a dichotomy of knowing/not knowing in the room that could make the MC2 students feel less capable, threatening the norm that all people were valuable contributors to the community.

Because of this threat, the teacher made two moves in the moment. First, she asked how they saw dividing an unknown height (q or c) by $\frac{1}{4}$ in the picture. No student readily saw that, so asking this question challenged all students and highlighted that everyone had something to learn, given that the class had a goal to always try to see their equations in their pictures. Second,

the teacher said they would continue to work on the issue of how to write division equations using multiplication, indicating the topic was not one that had to be resolved quickly. The first move was an attempt to bring students together by challenging them all together; the second move was an attempt to give students some space by acknowledging that learning takes time. In addition, during planning the research team decided to use heterogeneous groups for the next few episodes to avoid emphasizing differences. These moves helped maintain community for this class; we view them as a sign of how tenuous and problematic it can be to conduct whole class discussions across different thinkers, and how doing so can threaten community.

Episode 13, Experiment 3. However, practice 5 could also support DI, particularly when students were able to make some interpretations of each other's ideas. Episode 13 of Experiment 3 began with a whole class discussion of students' work on Two Unknowns Problems from the two prior episodes. Heather₂, Milo₂, and Symone₂ had worked on the **CS TP Hts Problem** (Figure 3); Yujeong₃ and Emmett₃ had worked on the **FSH Problem** (Figure 3), and Katrina₃ had worked on both problems. All six students were present.

The teacher began by projecting students' pictures and equations for the **CS TP Hts Problem**: $T \times 5 = C$ and $C \div 5 = T$, T = tomato plant height, C = cornstalk height. The teacher asked how students saw the division equation in the pictures. Yujeong₃ said, "if we divide the height of the cornstalk by five, we get the tomato plant's height." The teacher asked who could repeat this idea. After a 20-second pause, Heather₂ said, "Well if you, first..." She shook her head and said she did not know. The teacher thanked her for trying. Then Heather₂ said, "Well it's five equal parts in the cornstalk and if you divide them you get the tomato plant." Thus, she completed a thought that seemed challenging for her to articulate.

Then the teacher reminded students of a previous problem where they determined how many times 3 inches fit into 72 inches and notated that meaning as $72 \div 3$. She asked whether the meaning for division in $C \div 5 = T$ was this kind of “fitting into” meaning. Emmett₃, Yujeong₃, Heather₂, and Milo₂ all indicated yes, initially. Heather₂ covered her face with her hands, rubbed her temples, and said, “Because 5 goes into the cornstalk and one of them is the tomato plant.” The teacher asked if 5 was a measurement unit for the corn stalk height, as in 5 meant 5 inches. Heather₂ and Yujeong₃ both shook their heads no. Yujeong₃ explained that with a measurement meaning 5 would have to be inches, and they would be finding how many times 5 inches fits into the cornstalk height.

The teacher asked if they could write a measurement division equation for the **CS TP Hts Problem**. Students talked about this issue in small groups of three for about 2 minutes and came up with two equations, “ $1/5C = T$ ” and “ $C \div T = 5$.” In discussing $C \div T = 5$, the teacher asked if they could use “fitting into” language from earlier episodes to describe what was happening. Heather₂ immediately raised her hand: “Five fits into...” The teacher asked her to think about what was fitting into what in the situation. Heather₂ said, “Five fits into the cornstalk. Wait. The tomato plant fits into the cornstalk five times.” The teacher praised Heather₂ and asked other students to repeat this idea. The teacher noted that this fitting in idea might be useful in the work for that day, when all students would be working on problems with fractional relationships.

The teacher began to transition to group work. However, Emmett₃ asked if they could discuss the **FSH Problem**. The teacher agreed and asked if students could write a measurement division equation for this problem. Emmett₃ said, “Well, the sunflower fits into the fern 1 and $2/3$ times.” When asked how he could write that as an equation, he suggested “ $f \times 1 \frac{2}{3} = s$,” where f was the fern height and s the sunflower height (incorrect from our perspective). Then he

suggested “ $f \div 1.4 = s$ ” and then “ $f \div 1 \frac{2}{3} = s$,” which is a correct equation but not one with a measurement division meaning. Yujeong₃ suggested “ $f \div s = 5/3$.” Emmett₃ asked, “but 5/3 of what?” So, he did not appear to immediately agree with Yujeong₃. The teacher asked which of $f \div 1 \frac{2}{3} = s$ or $f \div s = 5/3$ showed Emmett₃’s idea about the sunflower height fitting into the fern height one whole time and 2/3 of a time. Heather immediately raised her hand. She selected $f \div s = 5/3$ because “the fern is five and the sunflower is three so it would be five-thirds.”

Heather₂’s reason for selecting $f \div s = 5/3$ does not indicate she saw the idea that Emmett₃ was voicing, that the sunflower height fit into the fern height $1 \frac{2}{3}$ times. In fact, based on our second-order model of her, most likely she was matching letters with numbers of parts: The fern height consisted of 5 parts and the sunflower height of 3 parts, so $f \div s$ should be $5 \div 3$, or $5/3$. However, we were still surprised that she tried to contribute at all to the discussion, given that we thought this problem to be quite far away from what she could work on productively. Instead, she seemed to want to contribute, and she made some interpretations of her peers’ work, although her understanding was quite different from Emmett₃’s and Yujeong₃’s. In sum, by discussing the MC3 students’ work on the **FSH Problem** she got exposed to ideas that she could make an interpretation of, even though she had not constructed concepts that allowed her to produce similar ideas on her own. Thus, we see conjecture 1 being upheld here.

This discussion is an example of tailoring instruction to student thinking by providing appropriate challenges. That is, focusing on the **CS TP Hts Problem** was fruitful for Heather₂ because it appeared to be provide plenty of challenge. Yet including the **FSH Problem** gave the MC3 students a stronger challenge and allowed Heather₂ to contribute in an unexpected way. Thus, this discussion upheld the norm that everyone was an active contributor, even to problems they had not directly worked on, thereby supporting the development of community.

Discussion

In the Findings section we responded to our first research question by identifying five teaching practices used to differentiate mathematics instruction for middle school students, and by demonstrating how the first two practices supported DI while the other three practices both supported and impeded DI. In this section we discuss the relationships between the five practices to respond to our second research question about a local theory of differentiating mathematics instruction for middle school students. Then we consider the contributions, limitations, and implications of the study.

Emergent Theory of Differentiating Mathematics Instruction

We propose relationships between the practices as our emergent theory of differentiating mathematics instruction for middle school students (Figure 9). We remind readers that we discussed the practices roughly in terms of how they would unfold in a lesson, in which planning practices occur before interacting practices. However, we came to understand practice 1, using research-based knowledge of students' mathematical thinking, and practice 3, inquiring responsively during group work, as mutually informing, and that together they led to practice 2, providing purposeful pathways, and to practice 5, conducting whole class discussions across different thinkers. We view the interplay of practices 1 and 3 to be at the heart of differentiating mathematics instruction because it is fundamentally about getting to know students' mathematical thinking while guided by ideas derived from prior research into students' mathematical thinking. Without getting to know students' thinking, there would be no reason to differentiate. Therefore, this interplay is a foundation for practices 2 and 5, which in turn, feedback to inform the interplay between practices 1 and 3. As discussed, practice 4, attending to

small group functioning, is a significant support for the interplay between practices 1 and 3 by making practice 3 possible, and we found that practices 4 and 5 also mutually shape each other.

Contributions

We start our discussion of contributions by identifying what this study is a case of for researchers and for teachers. For researchers, this study is a case of using second-order models of students' mathematical thinking to differentiate instruction. Such models provide teacher-researchers with constructs to interpret and intervene in students' activity, with the expectation that there will be some similarities between models so that students can be grouped together productively, as well as some differences that may require different interventions. The teacher's intervention in Milo₂'s activity at the start of his work on the **CS TP Hts Problem** is an example of the usefulness of models: The teacher's models of MC2 students had expanded over the prior two experiments to include that these students typically did not fix the stated multiplicative relationship between two unknowns. Thus, the teacher made an interpretation of Milo₂'s activity and an intervention, opening the possibility for him to iterate the smaller unknown to produce the larger unknown. We have found that MC3 students usually do not need this kind of intervention. Thus, models are valuable tools for DI. Although we do not have the expectation that classroom teachers would make such models, we suggest that using them to differentiate instruction provides a vision for DI in secondary mathematics classrooms.

For teachers, we view this study as a case in which attention to students' mathematical thinking and practices to differentiate instruction are intertwined. Furthermore, since the interplay between using research-based knowledge and inquiring responsively (practices 1 and 3) is the heart of DI, a teacher implementing DI with these practices must be an inquirer. As mentioned, teachers who follow CGI learn a research-based framework to help them determine

pathways to follow with their elementary school students. A critical feature of using a framework is inquiring into it as one interacts with students, using these interactions to deepen one's understanding of the framework and students together in order to adapt and extend the framework (Franke, et al., 2001; Steinberg, et al., 2004). Our study contributes to this literature on teacher-as-inquirer at the secondary level, where there are fewer such examples.

In relation to literature on DI, we propose that the interplay between using research-based knowledge and inquiring responsively operationalizes the goal of clarifying big ideas for students (Tomlinson, 2005) in mathematics. That is, inquiring into research-based knowledge and students' thinking together was a way for us to articulate our mathematical goals for students and develop our HLTs, thereby clarifying big ideas. This view means that clarifying big mathematical ideas in teaching is a dynamic process, marked by the on-going evolution of the teacher's first-order and second-order knowledge.

Another contribution to the DI literature has to do with norms. Literature on DI emphasizes the importance of individual autonomy as a norm (Laud, 2011; Tomlinson, 2005). Although this norm is important, we found that developing group autonomy was critical (practice 4). Proactively attending to group functioning in experiments 2 and 3 positively impacted our abilities to carry out all other practices, especially inquiring responsively (practice 3). Thus, the need to support students to develop group autonomy is a contribution to knowledge about DI.

In relation to mathematics education, providing choices and different pathways (practice 2) for students has been studied very little, and not at the secondary level (cf. Land, 2017). Because this practice allows teachers to tailor instruction while making features of student thinking visible, articulating practice 2 as a way to support students' diverse mathematical learning needs is a contribution to the field.

In contrast, conducting class discussions has received considerable attention in mathematics education (Jacobs & Spangler, 2017). Our study contributes to this literature because the teacher showcased differences without the expectation that everyone would think the same way. This position could devolve into an “anything goes” strategy-sharing stance that lacks mathematical substance (McClain & Cobb, 2001). We argue that our goals and HLTs from practice 1 were a guide to prevent that; we had both common and differentiated learning goals for MC2 and MC3 students. In addition, like the teacher in Staples’s (2007) study, we navigated differences while aiming to preserve a sense of connection around the larger goal of everyone learning more. For example, the teacher asked questions that were problematic for all, signaled that ideas did not have to get resolved in one class meeting, and purposefully varied groups.

Limitations

This study has several limitations with respect to understanding DI in mathematics classrooms. The setting of after school design experiments with small numbers of students did not address some realities of teaching, notably the pressure to cover topics in specific amounts of time, management of formative assessment from large numbers of students, and summative assessment and grading. For example, it is a significant feat to manage information about students’ thinking when a teacher is meeting 100 or more students each day. In addition, this study addressed DI for diversity in student thinking; it did not address other characteristics of students, such as their linguistic, cultural, and socioeconomic diversity (cf. Santamaria, 2009). Clearly, further research on DI in mathematics classrooms is needed, a topic we turn to next.

Implications for Teaching and Further Research

This study indicates that the five practices, as a web, could help teachers differentiate mathematics instruction for middle school students and guide professional development. To

begin, secondary teachers need access to research-based knowledge from which to build; in our view, they should not be expected to generate this knowledge as we did. Indeed, we want to underscore that we do not expect classroom teachers to make second-order models as we did. For that very reason, we articulated the first practice to be about using research-based knowledge in teaching; using second-order models as in this study is one example of this practice.

To be sure, some middle school mathematics curricular materials are research-based, such as the Connected Mathematics Project (CMP3) (Lappan et al., 2014). However, even materials like CMP3 do not provide teachers with detailed pathways for different thinkers that teachers could investigate as they inquire responsively into their students' thinking. We suggest that curricular materials need to be designed to target students operating with different multiplicative concepts. These materials need to be built on the ways of thinking of these students as represented in second-order models, and they need to support students to use their ways of thinking to learn. Some work has been done at the elementary school level toward this end (e.g., Hackenberg, Norton, & Wright, 2016; Wright, Stanger, Stafford, & Martland, 2014), but there is little for secondary school students.

In addition, further research is needed into the five practices with mathematics teachers in classrooms, investigating how they support teachers to differentiate instruction, how teachers learn them, how they influence students, and what needs to be added to them to address the limitations of this study. In particular, including other aspects of student diversity (linguistic, cultural, socioeconomic) in future studies is important. An open question is to what extent prospective teachers can learn these practices. Following Ball and Forzani (2009), who lay out possibilities for teacher education curricula built around practices, how might parts of the practices be learned by novices? We note that just because the practices are interconnected does

not mean that they cannot be worked on separately. In fact, it's important for both beginning and experienced teachers to take small steps in learning to differentiate mathematics instruction (Tomlinson, 2005), such as working on providing purposeful choices for students (part of practice 2). Yet we contend that seeing the practices as parts of a web will help teachers develop a larger vision for DI. Although considerable work is required to develop our emergent theory, we suggest that it is well worth it because of the need to support students to learn in heterogeneous mathematics classrooms. That is, if as a field we are serious about discontinuing practices of tracking that can be so harmful (NCTM 2018), we have an imperative to develop teaching practices that can truly support and celebrate diversity.

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